Towards Analytical Techniques for Optimizing Knowledge Acquisition, Processing, Propagation, and Use in Cyberinfrastructure

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Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

1. Introduction

- Knowledge-related processes are important: we rely on them when we drive, communicate, etc.
- Surprisingly, the very process of acquiring and propagating information is the least automated.
- At present, to decide on the best way to place sensors or propagate data, we mostly use numerical models.
- These models are very resource-consuming, rely on supercomputers, not ready for everyday applications.
- We therefore need analytical models – which would allow easier optimization and application.
- Developing such models is our main objective.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

2. Outline

- We describe analytical models for all stages of knowledge processing.
- We start with knowledge acquisition: optimal sensor placement for stationary and mobile sensors.
- We then deal with data and knowledge processing: how to best organize computing power and research teams.
- We deal with knowledge propagation and resulting knowledge enhancement; we analyze:
  - how early stages of idea propagation occur;
  - how to assess the initial knowledge level;
  - how to present the material and how to provide feedback.
- Finally, we analyze how knowledge is used.

Introduction

Outline

Sensor Placement: . . .

Data and Knowledge . . .

Knowledge . . .

Resulting Geometric . . .

How to Use the . . .

Future Work

3. Sensor Placement: Case Study

- Biological weapons are difficult and expensive to detect.
- Within a limited budget, we can afford a limited number of bio-weapon detector stations.
- It is therefore important to find the optimal locations for such stations.
- A natural idea is to place more detectors in the areas with more population.
- However, such a commonsense analysis does not tell us how many detectors to place where.
- To decide on the exact detector placement, we must formulate the problem in precise terms.

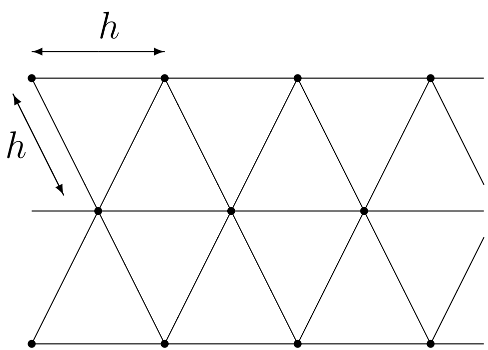
Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

4. Towards Precise Formulation of the Problem

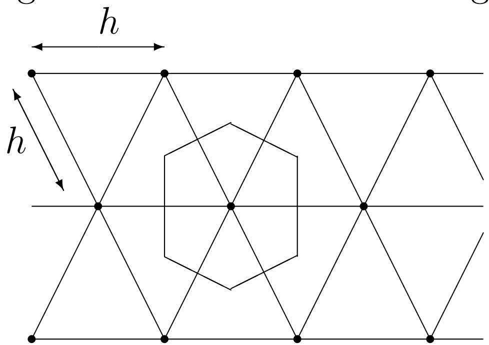
- The adversary's objective is to kill as many people as possible.
- Let $\rho(x)$ be a population density in the vicinity of the location x.
- Let N be the number of detectors that we can afford to place in the given territory.
- Let d_0 be the distance at which a station can detect an outbreak of a disease.
- Often, $d_0 = 0$ – we can only detect a disease when the sources of this disease reach the detecting station.
- We want to find $\rho_d(x)$ – the density of detector placement.
- We know that $\int \rho_d(x)\,dx = N$.

5. Optimal Placement of Sensors

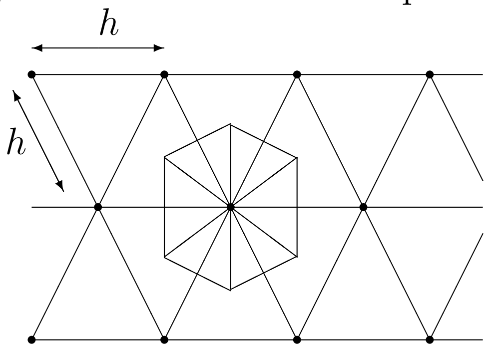
- We want to place the sensors in an area in such a way that
  - the largest distance D to a sensor
  - is as small as possible.
- It is known that the smallest such number is provided by an equilateral triangle grid:



For the equilateral triangle placement, points which are closest to a given detector forms a hexagonal area:



This hexagonal area consists of 6 equilateral triangles:



6. Optimal Placement of Sensors (cont-d)

- In each $\triangle$, the height $h/2$ is related to the side s by the formula $\dfrac{h}{2} = s \cdot \cos(60^\circ) = s \cdot \dfrac{\sqrt{3}}{2}$, hence $s = h \cdot \dfrac{\sqrt{3}}{3}$.
- Thus, the area A_t of each triangle is equal to

$$A_t = \frac{1}{2} \cdot s \cdot \frac{h}{2} = \frac{1}{2} \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{2} \cdot h^2 = \frac{\sqrt{3}}{12} \cdot h^2.$$

- So, the area A_s of the whole set is equal to 6 times the triangle area: $A_s = 6 \cdot A_t = \dfrac{\sqrt{3}}{2} \cdot h^2$.
- In a region of area A, there are $A \cdot \rho_d(x)$ sensors, they cover area $(A \cdot \rho_d(x)) \cdot A_s$.
- The condition $A = (A \cdot \rho_d(x)) \cdot A_s = (A \cdot \rho_d(x)) \cdot \dfrac{\sqrt{3}}{2} \cdot h^2$ implies that $h = \dfrac{c_0}{\sqrt{\rho_d(x)}}$, with $c_0 \stackrel{\text{def}}{=} \sqrt{\dfrac{2}{\sqrt{3}}}$.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

7. Estimating the Effect of Sensor Placement

- The adversary places the bio-weapon at a location which is the farthest away from the detectors.
- This way, it will take the longest time to be detected.
- For the grid placement, this location is at one of the vertices of the hexagonal zone.
- At these vertices, the distance from each neighboring detector is equal to $s = h \cdot \dfrac{\sqrt{3}}{3}$.
- By know that $h = \dfrac{c_0}{\sqrt{\rho_d(x)}}$, so $s = \dfrac{c_1}{\sqrt{\rho_d(x)}}$, with
$$c_1 = \frac{\sqrt{3}}{3} \cdot c_0 = \frac{\sqrt[4]{3} \cdot \sqrt{2}}{3}.$$
- Once the bio-weapon is placed, it starts spreading until it reaches the distance d_0 from the detector.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

8. Effect of Sensor Placement (cont-d)

- The bio-weapon is placed at a distance $s = \dfrac{c_1}{\sqrt{\rho_d(x)}}$ from the nearest sensor.
- Once the bio-weapon is placed, it starts spreading until it reaches the distance d_0 from the detector.
- In other words, it spreads for the distance $s - d_0$.
- During this spread, the disease covers the circle of radius $s - d_0$ and area $\pi \cdot (s - d_0)^2$.
- The number of affected people $n(x)$ is equal to:

$$n(x) = \pi \cdot (s - d_0)^2 \cdot \rho(x) = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0\right)^2 \cdot \rho(x).$$

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

9. Precise Formulation of the Problem

- For each location x, the number of affected people $n(x)$ is equal to:

$$n(x) = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x).$$

- The adversary will select a location x for which this number $n(x)$ is the largest possible:

$$n = \max_x \left(\pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x) \right).$$

- Resulting problem:
  - given population density $\rho(x)$, detection distance d_0, and number of sensors N,
  - find a function $\rho_d(x)$ that minimizes the above expression n under the constraint $\int \rho_d(x)\,dx = N$.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

10. Main Lemma

- *Reminder:* we want to minimize the worst-case damage $n = \max_x n(x)$.
- *Lemma:* for the optimal sensor selection, $n(x) = \text{const}$.
- *Proof by contradiction:* let $n(x) < n$ for some x; then:
  - we can slightly increase the detector density at the locations where $n(x) = n$,
  - at the expense of slightly decreasing the location density at locations where $n(x) < n$;
  - as a result, the overall maximum $n = \max_x n(x)$ will decrease;
  - but we assumed that n is the smallest possible.
- *Thus:* $n(x) = \text{const}$; let us denote this constant by n_0.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

11. Towards the Solution of the Problem

- We have proved that $n(x) = \text{const} = n_0$, i.e., that

$$n_0 = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x).$$

- Straightforward algebraic transformations lead to:

$$\rho_d(x) = \frac{2 \cdot \sqrt{3}}{9} \cdot \frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}} \right)^2}.$$

- The value c_2 must be determined from the equation

$$\int \rho_d(x)\, dx = N.$$

- Thus, we arrive at the following solution.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

12. Solution

- *General case:* the optimal detector location is characterized by the detector density

$$\rho_d(x) = \frac{2\cdot\sqrt{3}}{9}\cdot\frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}}\right)^2}.$$

- Here the parameter c_2 must be determined from the equation $\int \frac{2\cdot\sqrt{3}}{9}\cdot\frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}}\right)^2}\,dx = N.$

- *Case of $d_0 = 0$:* in this case, the formula for $\rho_d(x)$ takes a simplified form $\rho_d(x) = C\cdot\rho(x)$ for some constant C.

- In this case, from the constraint, we get:

$$\rho_d(x) = \frac{N}{N_p}\cdot\rho(x),\ \text{ where } N_p \text{ is the total population.}$$

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

13. Towards More Relevant Objective Functions

- We assumed that the adversary wants to maximize the number $\int \rho(x)\,dx$ of people affected by the bio-weapon.
- The actual adversary's objective function may differ from this simplified objective function.
- For example, the adversary may take into account that different locations have different publicity potential.
- In this case, the adversary maximizes the weighted value $\int_A \widetilde{\rho}(x)\,dx$, where $\widetilde{\rho}(x) \stackrel{\text{def}}{=} w(x) \cdot \rho(x)$.
- Here, $w(x)$ is the importance of the location x.
- From the math. viewpoint, the problem is the same – w/"effective population density" $\widetilde{\rho}(x)$ instead of $\rho(x)$.
- Thus, if we know $w(x)$, we can find the optimal detector density $\rho_d(x)$ from the above formulas.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

14. How Temperatures etc. Change from One Spatial Location to Another: A Model

- Each environmental characteristic q changes from one spatial location to another.
- A large part of this change is unpredictable (i.e., random).
- A reasonable value to describe the random component of the difference $q(x)-q(x')$ is the variance

$$V(x,x') \stackrel{\text{def}}{=} E[((q(x)-E[q(x)])-(q(x')-E[q(x')]))^2].$$

- *Comment:* we assume that averages are equal.
- Locally, processes should not change much with shift $x \to x+s$: $V(x+s,x'+s)=V(x,x')$.
- For $s=-x'$, we get $V(x,x')=C(x-x')$ for

$$C(x) \stackrel{\text{def}}{=} V(x,0).$$

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

15. A Model (cont-d)

- In general, the further away the points x and x', the larger the difference $C(x - x')$.
- In the isotropic case, $C(x - x')$ depends only on the distance $D = |x - x'|^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2$.
- It is reasonable to consider a scale-invariant dependence $C(x) = A \cdot D^\alpha$.
- In practice, we may have more changes in one direction and less change in another direction.
- E.g., 1 km in x is approximately the same change as 2 km in y.
- The change can also be mostly in some other direction, not just x- and y-directions.
- Thus, in general, in appropriate coordinates (u, v), we have $C = A \cdot D^\alpha$ for $D = (u - u')^2 + (v - v')^2$.

16. Model: Final Formulas

- In general, $C = A \cdot D^\alpha$, for $D = (u - u')^2 + (v - v')^2$ in appropriate coordinates (u, v).
- In the original coordinates x_1 and x_2, we get:

$$C(x - x') = A \cdot D^\alpha, \text{ where}$$

$$D = \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij} \cdot (x_i - x'_i) \cdot (x_j - x'_j) =$$

$$g_{11} \cdot (x_1 - x'_1)^2 + 2g_{12} \cdot (x_1 - x'_1) \cdot (x_2 - x'_2) + g_{22} \cdot (x_2 - x'_2)^2.$$

- From the computational viewpoint, we can include A into g_{ij} if we replace g_{ij} with $A^{1/\alpha} \cdot g_{ij}$, then

$$C(x - x') =$$

$$\left(g_{11} \cdot (x_1 - x'_1)^2 + 2g_{12} \cdot (x_1 - x'_1) \cdot (x_2 - x'_2) + g_{22} \cdot (x_2 - x'_2)^2\right)^\alpha$$

- We can use these formulas to find the optimal sensor locations.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

17. Optimal Use of Mobile Sensors: Case Study

- Remote areas of international borders are used by the adversaries: to smuggle drugs, to bring in weapons.
- It is therefore desirable to patrol the border, to minimize such actions.
- It is not possible to effectively man every single segment of the border.
- It is therefore necessary to rely on other types of surveillance.
- Unmanned Aerial Vehicles (UAVs):
  - from every location along the border, they provide an overview of a large area, and
  - they can move fast, w/o being slowed down by clogged roads or rough terrain.
- Question: what is the optimal trajectory for these UAVs?

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

18. How to Describe Possible UAV Patrolling Strategies

- Let us assume that the time between two consequent overflies is smaller the time needed to cross the border.
- Ideally, such a UAV can detect all adversaries.
- In reality, a fast flying UAV can miss the adversary.
- We need to minimize the effect of this miss.
- The faster the UAV goes, the less time it looks, the more probable that it will miss the adversary.
- Thus, the velocity $v(x)$ is very important.
- By a patrolling strategy, we will mean a f-n $v(x)$ describing how fast the UAV flies at different locations x.

19. Constraints on Possible Patrolling Strategies

1) The time between two consequent overflies should be smaller the time T needed to cross the border:

   – the time during which a UAV passes from the location x to the location $x+\Delta x$ is equal to $\Delta t = \dfrac{\Delta x}{v(x)}$;

   – thus, the overall flight time is equal to the sum of these times:
   $$T = \int \frac{dx}{v(x)}.$$

2) UAV has the largest possible velocity V, so we must have $v(x) \le V$ for all x.

It is convenient to use the value $s(x) \stackrel{\text{def}}{=} \dfrac{1}{v(x)}$ called *slowness*, so

$$T = \int s(x)\,dx; \quad s(x) \ge S \left(\stackrel{\text{def}}{=} \frac{1}{V} \right).$$

Introduction
Outline
Sensor Placement: ...
Data and Knowledge ...
Knowledge ...
Resulting Geometric ...
How to Use the ...
Future Work

20. Simplification of the Constraints

- Since $s(x) \geq S$, the value $s(x)$ can be represented as $S + \Delta s(x)$, where $\Delta s(x) \stackrel{\text{def}}{=} s(x) - S$.
- The new unknown function satisfies the simpler constraint $\Delta s(x) \geq 0$.
- In terms of $\Delta s(x)$, the requirement that the overall time be equal to T has a form $T = S \cdot L + \int \Delta s(x)\, dx$.
- This is equivalent to:

$$T_0 = \int \Delta s(x)\, dx, \text{ where:}$$

  - L is the total length of the piece of the border that we are defending, and
  - $T_0 \stackrel{\text{def}}{=} T - S \cdot L$.

Introduction
Outline
Sensor Placement: ...
Data and Knowledge ...
Knowledge ...
Resulting Geometric ...
How to Use the ...
Future Work

21. Detection at Crossing Point x

- Let h be the width of the border zone from which an adversary (A) is visible.
- Then, the UAV can potentially detect A during the time $h/v(x) = h \cdot s(x)$.
- So, the UAV takes $(h \cdot s(x))/\Delta t$ photos, where Δt is the time per photo.
- Let p_1 be the probability that one photo misses A.
- It is reasonable to assume that different detection errors are independent.
- Then, the probability $p(x)$ that A is not detected is $p_1^{(h \cdot s(x))/\Delta t}$, i.e., $p(x) = \exp(-k \cdot s(x))$, where:

$$k \stackrel{\text{def}}{=} \frac{2h}{\Delta t} \cdot |\ln(p_1)|.$$

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

22. Strategy Selected by the Adversary

- Let $w(x)$ denote the utility of the adversary succeeding in crossing the border at location x.
- Let us first assume that we know $w(x)$ for every x.
- According to decision theory, the adversary will select a location x with the largest expected utility

$$u(x) = p(x) \cdot w(x) = \exp(-k \cdot s(x)) \cdot w(x).$$

- Thus, for each slowness function $s(x)$, the adversary's gain $G(s)$ is equal to

$$G(s) = \max_x u(x) = \max_x \left[\exp(-k \cdot s(x)) \cdot w(x)\right].$$

- We need to select a strategy $s(x)$ for which the gain $G(s)$ is the smallest possible.

$$G(s) = \max_x u(x) = \max_x \left[\exp(-k \cdot s(x)) \cdot w(x)\right] \to \min_{s(x)}.$$

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

23. Towards an Optimal Strategy for Patrolling the Border

- Let x_m be the location at which the utility $u(x) = \exp(-k \cdot s(x)) \cdot w(x)$ attains its largest possible value.
- If we have a point x_0 s.t. $u(x_0) < u(x_m)$ and $s(x_0) > S$:
  - we can slightly decrease the slowness $s(x_0)$ at the vicinity of x_0 (i.e., go faster in this vicinity) and
  - use the resulting time to slow down (i.e., to go slower) at all locations x at which $u(x) = u(x_m)$.
- As a result, we slightly decrease the value

$$u(x_m) = \exp(-k \cdot s(x_m)) \cdot w(x_m).$$

- At x_0, we still have $u(x_0) < u(x_m)$.
- So, the overall gain $G(s)$ decreases.
- Thus, when the adversary's gain is minimized, we get

$$u(x) = u_0 = \text{const whenever } s(x) > S.$$

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

24. Towards an Optimal Strategy (cont-d)

- *Reminder:* for the optimal strategy,

  $$u(x) = w(x) \cdot \exp(-k \cdot s(x)) = u_0 \text{ whenever } s(x) > S.$$

- So, $\exp(-k \cdot s(x)) = \dfrac{u_0}{w(x)}$, hence

  $$s(x) = \frac{1}{k} \cdot (\ln(w(x)) - \ln(u_0)) \text{ and } \Delta s(x) = \frac{1}{k} \cdot \ln(w(x)) - \Delta_0.$$

- Here, $\Delta_0 \stackrel{\text{def}}{=} \dfrac{1}{k} \cdot \ln(u_0) - S$.
- When $s(x)$ gets to $s(x) = S$ and $\Delta s(x) = 0$, we get $\Delta s(x) = 0$.
- Thus, we conclude that

  $$\Delta s(x) = \max\left(\frac{1}{k} \cdot \ln(w(x)) - \Delta_0, 0\right).$$

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

25. An Optimal Strategy: Algorithm

- *Reminder:* for some Δ_0, the optimal strategy has the form

$$\Delta s(x) = \max\left(\frac{1}{k}\cdot \ln(w(x)) - \Delta_0, 0\right).$$

- *How to find Δ_0:* from the condition that

$$\int \Delta s(x)\,dx =$$

$$\int \max\left(\frac{1}{k}\cdot \ln(w(x)) - \Delta_0, 0\right)\,dx = T_0.$$

- *Easy to check:* the above integral monotonically decreases with Δ_0.

- *Conclusion:* we can use bisection to find the appropriate value Δ_0.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

26. Efficient Algorithms for Optimizing Sensor Use: Case Study of Security Problems

- In this section, we analyze the problem of designing efficient algorithms for optimizing resource allocations.
- Case study: protection of critical infrastructure from terrorist attacks, computer network security, etc.
- Previously known algorithms for optimal resource allocation required quadratic time $O(n^2)$.
- We develop new algorithms which require time

$$O(n \cdot \log(n)) \ll O(n^2).$$

- In important special cases, this algorithm runs even faster, in linear time $O(n)$.

27. Data and Knowledge Processing: How to Best Organize Computing and Human Resources

- Once the data is collected, we need to process this data.
- For processing, we need computing power, and we need human resources.
- In both cases, we need to come up with optimal resource allocation.
- In Section 3.1, we come up with the optimal allocation formulas for computing resources.
- In Section 3.2, we come up with the optimal allocation formulas for human resources.
- The corresponding mathematics is similar to the optimal distribution of sensors.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

28. Knowledge Propagation and Resulting Knowledge Enhancement

- Once we have transformed data into knowledge, we need to propagate this knowledge.
- For that, we first need to motivate people to learn the new knowledge.
- To ensure this, we analyze the process of knowledge propagation.
- On early stages, the number of knowledgeable people grows as a power law t^α.
- In Section 4.1, we provide a theoretical explanation for this empirical fact.
- Once a person is interested, we assess how much he/she knows, and how to best teach the material.
- This is covered in Sections 4.2-4.5.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

29. Assessing the Initial Knowledge Level

- Computers enable us to provide individualized learning, at a pace tailored to each student.
- In order to start the learning process, it is important to find out the current level of the student's knowledge.
- Usually, such placement tests use a sequence of N problems of increasing complexity.
- If a student is able to solve a problem, the system generates a more complex one.
- If a student cannot solve a problem, the system generates an easier one, etc.
- Once we find the exact level of student's knowledge, the actual learning starts.
- It is desirable to get to actual leaning as soon as possible, i.e., to minimize the # of placement problems.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

30. Bisection – Optimal Search Procedure

- At each stage, we have:
  - the largest level i at which a student can solve, &
  - the smallest level j at which s/he cannot.
- Initially, $i = 0$ (trivial), $j = N + 1$ (very tough).
- If $j = i + 1$, we found the student's level of knowledge.
- If $j > i + 1$, give a problem on level $m \stackrel{\text{def}}{=} (i + j)/2$:
  - if the student solved it, increase i to m;
  - else decrease j to m.
- In both cases, the interval $[i, j]$ is decreased by half.
- In s steps, we decrease the interval $[0, N + 1]$ to width $(N + 1) \cdot 2^{-s}$.
- In $s = \lceil \log_2(N + 1) \rceil$ steps, we get the interval of width ≤ 1, so the problem is solved.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

31. Need to Account for Discouragement

- Every time a student is unable to solve a problem, he/she gets discouraged.
- In bisection, a student whose level is 0 will get $\approx \log_2(N+1)$ negative feedbacks.
- For positive answers, the student simply gets tired.
- For negative answers, the student also gets stressed and frustrated.
- If we count an effect of a positive answer as one, then the effect of a negative answer is $w > 1$.
- The value w can be individually determined.
- We need a testing scheme that minimizes the worst-case overall effect.

32. Analysis of the Problem

- We have $x = N + 1$ possible levels of knowledge.
- Let $e(x)$ denote the smallest possible effect needed to find out the student's knowledge level.
- We ask a student to solve a problem of some level n.
- If s/he solved it (effect $= 1$), we have $x - n$ possible levels $n, \ldots, N$.
- The effect of finding this level is $e(x - n)$, so overall effect is $1 + e(x - n)$.
- If s/he didn't (effect w), his/her level is between 0 and n, so we need effect $e(n)$, with overall effect $w + e(n)$.
- Overall worst-case effect is $\max(1+e(x-n), w+e(n))$.
- In the optimal test, we select n for which this effect is the smallest, so $e(x) = \min\limits_{1 \le n < x} \max(1+e(x-n), w+e(n))$.

33. Resulting Algorithm

- For $x = 1$, i.e., for $N = 0$, we have $e(1) = 0$.
- We know that $e(x) = \min\limits_{1 \le n < x} \max(1 + e(x - n), w + e(n))$.
- We can use this formula to sequentially compute the values $e(2)$, $e(3)$, $\ldots$, $e(N + 1)$.
- We also compute the corresponding minimizing values $n(2)$, $n(3)$, $\ldots$, $n(N + 1)$.
- Initially, $i = 0$ and $j = N + 1$.
- At each iteration, we ask to solve a problem at level $m = i + n(j - i)$:
  - if the student succeeds, we replace i with m;
  - else we replace j with m.
- We stop when $j = i + 1$; this means that the student's level is i.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

34. Example 1: $N = 3$, $w = 3$

- Here, $e(1) = 0$.
- When $x = 2$, the only possible value for n is $n = 1$, so

$$e(2) = \min_{1 \le n < 2} \{\max\{1 + e(2 - n), 3 + e(n)\}\} =$$

$$\max\{1 + e(1), 3 + e(1)\} = \max\{1, 3\} = 3.$$

- Here, $e(2) = 3$, and $n(2) = 1$.
- To find $e(3)$, we must compare two different values $n = 1$ and $n = 2$:

$$e(3) = \min_{1 \le n < 3} \{\max\{1 + e(3 - n)), 3 + e(n)\}\} =$$

$$\min\{\max\{1+e(2), 3+e(1)\}, \max\{1+e(1), 3+e(2)\}\} =$$

$$\min\{\max\{4, 3\}, \max\{1, 6\}\} = \min\{4, 6\} = 4.$$

- Here, min is attained when $n = 1$, so $n(3) = 1$.

35. Example 1: $N = 3$, $w = 3$ (cont-d)

- To find $e(4)$, we must consider three possible values $n = 1$, $n = 2$, and $n = 3$, so

$$e(4) = \min_{1 \le n < 4} \{\max\{1 + e(4 - n), 3 + e(n))\}\} =$$

$$\min\{\max\{1 + e(3), 3 + e(1)\}, \max\{1 + e(2), 3 + e(2)\},$$

$$\max\{1 + e(1), 3 + e(3)\}\} =$$

$$\min\{\max\{5, 3\}, \max\{4, 6\}, \max\{1, 7\}\} =$$

$$\min\{5, 6, 7\} = 5.$$

- Here, min is attained when $n = 1$, so $n(4) = 1$.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

36. Example 1: Resulting Procedure

- First, $i = 0$ and $j = 4$, so we ask a student to solve a problem at level $i + n(j - i) = 0 + n(4) = 1$.
- If the student fails level 1, his/her level is 0.
- If s/he succeeds at level 1, we set $i = 1$, and we assign a problem of level $1 + n(3) = 2$.
- If the student fails level 2, his/her level is 1.
- If s/he succeeds at level 2, we set $i = 2$, and we assign a problem of level $2 + n(3) = 3$.
- If the student fails level 3, his/her level is 2.
- If s/he succeeds at level 3, his/her level is 3.
- We can see that this is the most cautious scheme, when each student has at most one negative experience.

37. Example 2: $N = 3$ and $w = 1.5$

- We take $e(1) = 0$.
- When $x = 2$, then

$$e(2) = \min_{1 \le n < 2} \{\max\{1 + e(2-n), 3 + e(n)\}\} =$$

$$\max\{1 + e(1), 1.5 + e(1)\} = \max\{1, 1.5\} = 1.5.$$

- Here, $e(2) = 1.5$, and $n(2) = 1$.
- To find $e(3)$, we must compare two different values $n = 1$ and $n = 2$:

$$e(3) = \min_{1 \le n < 3} \{\max\{1 + e(3-n)), 1.5 + e(n)\}\} =$$

$$\min\{\max\{1+e(2), 1.5+e(1)\}, \max\{1+e(1), 1.5+e(2)\}\} =$$

$$\min\{\max\{2.5, 1.5\}, \max\{1, 3\}\} = \min\{2.5, 3\} = 2.5.$$

- Here, min is attained when $n = 1$, so $n(3) = 1$.

38. Example 2: $N = 3$ and $w = 1.5$ (cont-d)

- To find $e(4)$, we must consider three possible values $n = 1$, $n = 2$, and $n = 3$, so

$$e(4) = \min_{1 \le n < 4} \{\max\{1 + e(4-n), 1.5 + e(n))\}\} =$$

$$\min\{\max\{1+e(3), 1.5+e(1)\}, \max\{1+e(2), 1.5+e(2)\},$$

$$\max\{1 + e(1), 1.5 + e(3)\}\} =$$

$$\min\{\max\{3.5, 1.5\}, \max\{2.5, 3\}, \max\{1, 4\}\} =$$

$$\min\{3.5, 3, 4\} = 3.$$

- Here, min is attained when $n = 2$, so $n(4) = 2$.

39. Example 2: Resulting Procedure

- First, $i = 0$ and $j = 4$, so we ask a student to solve a problem at level $i + n(j - i) = 0 + n(4) = 2$.
- If the student fails level 2, we set $j = 2$, and we assign a problem of level $0 + n(2) = 1$:
  - if the student fails level 1, his/her level is 0;
  - if s/he succeeds at level 1, his/her level is 1.
- If s/he succeeds at level 2, we set $i = 2$, and we assign a problem at level $2 + n(2) = 3$:
  - if the student fails level 3, his/her level is 2;
  - if s/he succeeds at level 3, his/her level is 3.
- We can see that in this case, the optimal testing scheme is bisection.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

40. A Faster Algorithm May Be Needed

- For each n from 1 to N, we need to compare n different values.
- So, the total number of computational steps is proportional to $1 + 2 + \ldots + N = O(N^2)$.
- When N is large, N^2 may be too large.
- In some applications, the computation of the optimal testing scheme may takes too long.
- For this case, we have developed a faster algorithm for producing a testing scheme.
- The disadvantage of this algorithm is that it is only asymptotically optimal.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

41. A Faster Algorithm for Generating an Asymptotically Optimal Testing Scheme

- First, we find the real number $\alpha \in [0,1]$ for which $\alpha + \alpha^w = 1$.
- This value α can be obtained, e.g., by applying bisection to the equation $\alpha + \alpha^w = 1$.
- At each iteration, once we know bounds i and j, we ask the student to solve a problem at the level
$$m = \lfloor \alpha \cdot i + (1 - \alpha) \cdot j \rfloor.$$
- This algorithm is similar to bisection, expect that bisection corresponds to $\alpha = 0.5$.
- This makes sense, since for $w = 1$, the equation for α takes the form $2\alpha = 1$, hence $\alpha = 0.5$.
- For $w = 2$, the solution to the equation $\alpha + \alpha^2 = 1$ is the well-known golden ratio $\alpha = \dfrac{\sqrt{5} - 1}{2} \approx 0.618$.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

42. Towards Optimal Teaching

- One of the main objectives of a course – calculus, physics, etc. – is to help students understand its main concepts.
- Of course, it is also desirable that the students learn the corresponding methods and algorithms.
- However, understanding is the primary goal.
- If a student does not remember a formula by heart, she can look it up.
- However:
  - if a student does not have a good understanding of what, for example, is a derivative,
  - then even if this student remembers some formulas, he will not be able to decide which formula to apply.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

43. How to Gauge Student Understanding

- To properly gauge student's understanding, several disciplines have developed *concept inventories*.
- These are sets of important basic concepts and questions testing the students' understanding.
- The first such Force Concept Inventory (FCI) was developed to gauge the students' understanding of forces.
- A student's degree of understanding is measured by the percentage of the questions that are answered correctly.
- The class's degree of understanding is measured by averaging the students' degrees.
- An ideal situation is when everyone has a perfect 100% understanding; in this case, the average score is 100%.
- In practice, the average score is smaller than 100%.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

44. How to Compare Different Teaching Techniques

- We can measure the average score μ_0 before the class and the average score μ_f after the class.
- Ideally, the whole difference $100 - \mu_0$ disappears, i.e., the students' score goes from μ_0 to $\mu_f = 100$.
- In practice, of course, the students' gain $\mu_f - \mu_0$ is somewhat smaller than the ideal gain $100 - \mu_0$.
- It is reasonable to measure the success of a teaching method by which portion of the ideal gain is covered:

$$g \stackrel{\text{def}}{=} \frac{\mu_f - \mu_0}{100 - \mu_0}.$$

Introduction
Outline
Sensor Placement: ...
Data and Knowledge ...
Knowledge ...
Resulting Geometric ...
How to Use the ...
Future Work

45. Empirical Results

- It turns out that the gain g does not depend on the initial level μ_0, on the textbook used, or on the teacher.
- Only one factor determines the value g: the absence or presence of immediate feedback.
- In traditionally taught classes,
  - where the students get their major feedback only after their first midterm exam,
  - the average gain is $g \approx 0.23$.
- For the classes with an immediate feedback, the average gain is twice larger: $g \approx 0.48$.
- In this talk, we provide a possible geometric explanation for this doubling of the learning rate.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

46. Why Geometry

- Learning means changing the state of a student.
- At each moment of time, the state can be described by the scores $x_1, \ldots, x_n$ on different tests.
- Each such state can be naturally represented as a point $(x_1, \ldots, x_n)$ in the n-dimensional space.
- In the starting state S, the student does not know the material.
- The desired state D describes the situation when a student has the desired knowledge.
- When a student learns, the student's state of knowledge changes continuously.
- It forms a (continuous) trajectory γ which starts at the starting state S and ends up at the desired state D.

Introduction
Outline
Sensor Placement: ...
Data and Knowledge ...
Knowledge ...
Resulting Geometric ...
How to Use the ...
Future Work

47. First Simplifying Assumption: All Students Learn at the Same Rate

- Some students learn faster, others learn slower.
- The above empirical fact, however, is not about their *individual* learning rates.
- It is about the *average* rates of student learning, averaged over all kinds of students.
- From this viewpoint, it makes sense to assume that all the students have the same average learning rate.
- In geometric terms, this means that the leaning time is proportional to the length of the corresponding curve γ.
- We thus need to show that learning trajectories corr. to immediate feedback are, on average, twice shorter.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

48. Second Simplifying Assumption: the Shape of the Learning Trajectories

- At first, a student has misconceptions about physics or calculus, which lead him in a wrong direction.
- We can thus assume that at first, a student moves in a random direction.
- After the feedback, the student corrects his/her trajectory.
- In the case of immediate feedback, this correction comes right away, so the students goes in the right direction.
- In the traditional learning, with a midterm correction:
  - a student first follows a straight line of length $d/2$ which goes in a random direction,
  - and then takes a straight line to the midpoint M.
- Then, a student goes from M to the destination D.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

49. 3rd Simplifying Assumption: 1-D State Space

- We can think of different numerical characteristics describing different aspects of student knowledge.
- In practice, to characterize the student's knowledge, we use a single number – the overall grade for the course.
- It is therefore reasonable to assume that the state of a student is characterized by only one parameter x_1.
- In case of immediate feedback, the learning trajectory has length d.
- To make a comparison, we must estimate the length of a trajectory corresponding to the traditional learning.
- This trajectory consists of two similar parts: connecting S and M and connecting M and D.
- To estimate the total average length, we can thus estimate the average length from S to M and double it.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

50. Analysis: Case of Traditional Leaning

- A student initially goes either in the correct direction or in the opposite (wrong) direction.
- Randomly means that both directions occur with equal probability $1/2$.
- If the student moves in the right direction, she gets exactly into the desired midpoint M.
- In this case, the length of the S-to-M part of the trajectory is exactly $d/2$.
- If the student starts in the wrong direction, he ends up at a point at distance $d/2$ – on the wrong side of S.
- Getting back to M then means first going back to S and then going from S to M.
- The overall length of this trajectory is thus $3d/2$.

51. Resulting Geometric Explanation

- Here:
  - with probability $1/2$, the length is $d/2$;
  - with probability $1/2$, the length is $3d/2$.
- So, the average length of the S-to-M part of the learning trajectory is equal to

$$\frac{1}{2} \cdot \frac{d}{2} + \frac{1}{2} \cdot \frac{3d}{2} = d.$$

- The average length of the whole trajectory is double that, i.e., $2d$.
- This average length is twice larger than the length d corresponding to immediate feedback.
- This explains why immediate feedback makes learning, on average, twice faster.

52. How to Use the Resulting Knowledge: Case Study

- How can we use the acquired knowledge?
- In many practical situations, we have a well-defined problem, with a clear well-formulated objective.
- Such problems are typical in engineering:
  - we want a bridge which can withstand a given load,
  - we want a car with a given fuel efficiency, etc.
- However, in many practical situations, it is important to also take into account subjective user preferences.
- This subjective aspect of decision making is known as *Kansei engineering*.

53. Need to Select Designs

- Different people have different preferences.
- Thus, to satisfy customers, we must produce several different designs:
  - a car company produces cars of several different designs,
  - a furniture company produces chairs of several different designs, etc.
- The creation of each new design is often very expensive and time-consuming.
- As a result, the number of new designs is usually limited.
- Once we know what customers want and how many designs we can afford, how to select these designs?

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

54. Results

- In Chapter 5, we come up with analytical formulas for optimal design selection.
- These formulas are similar to formulas of optimal sensor allocation.
- Each design can be characterized by an n-dimensional vector $x = (x_1, \ldots, x_n)$.
- Each user has his/her own ideal design x.
- For many users, we have a density $\rho_u(x)$ corr. to different designs.
- We provide an analytical formula describing the optimal design density $\rho_m(x)$.

55. Future Work: Main Theoretical Activity

- Recently, a seminal book appeared *Social Physics* by A. Pentland from MIT.
- This book describes the successful results of using models to enhance knowledge propagation.
- This book describes many well-justified results.
- It also describes interesting empirical observations for which no theoretical explanations are available.
- Our plan is to look into these observations and results and see if some of them can be theoretically explained.

56. Future Work: Auxiliary Theoretical Activity

- Petland's book views knowledge propagation from the viewpoint of mathematical optimization.
- We thus try to find the best values of the parameters of the corresponding knowledge propagation process.
- In this model, all the decisions are centralized.
- Educational practice shows that often, efficiency can be drastically improved by decentralization.
- Specifically, we allow teachers – and students – to select different ways of propagating knowledge.
- There have been several empirical studies of this phenomenon.
- We plan to look for a theoretical explanation for the known empirical results.

Introduction
Outline
Sensor Placement: . . .
Data and Knowledge . . .
Knowledge . . .
Resulting Geometric . . .
How to Use the . . .
Future Work

57. Future Work: Practical Applications

- The ultimate goal of the theoretical research is to enhance actual knowledge propagation.
- As part of our research, we have already developed some practical recommendations.
- We plan to test these recommendations on the actual processes of teaching and knowledge propagation.
- In particular, we plan to test them on cyberinfrastructure-related data acquisition, processing, and propagation.
- These applications are what motivated our research.
- We thus hope that our recommendations will be useful for cyberinfrastructure-related applications.

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Introduction
Outline
Sensor Placement: ...
Data and Knowledge ...
Knowledge ...
Resulting Geometric ...
How to Use the ...
Future Work