Differentiation Beyond Traditional Definitions: Case Studies of Application-Motivated Extensions

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1. Traditional Differentiation

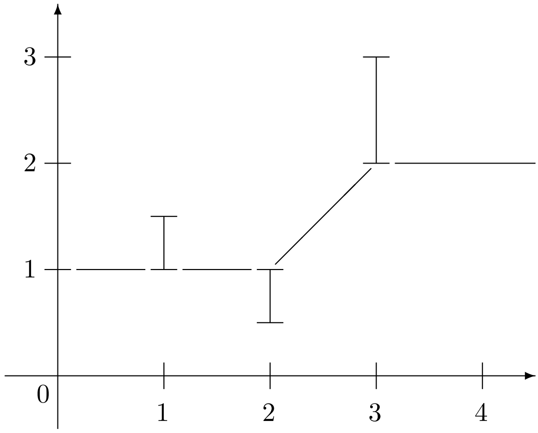
- *When:* differentiation was originally invented by Newton and Leibniz.
- *Why:* differentiation was invented:
  - to find local and global maxima and minima, and
  - to describe dynamics of real-life systems (by using differential equations).
- *How:* in the traditional calculus definitions, we have:
  - an exactly known function $f(x_1, \ldots, x_n)$
  - from real numbers to real numbers.

2. Limitations of Traditional Differentiation

- In real life problems, the situation is often more complicated that for traditional differentiation.
- We often know only a few values of the (unknown) function, and even these values come from measurement and are thus known with (*interval*) uncertainty.
- In some problems, e.g., in design, we want to *find* a function or *a set* that would optimize the desired objective.
- In some problems, we would like to describe the dynamics of a system whose parameters are *not real numbers*.

3. Interval Uncertainty

- *Example:* a 1-D landscape is a dependence of the altitude y on the coordinate x:
  - low values of the slope dy/dx correspond to a plain,
  - high values to steep mountains, and
  - medium values to a hilly terrain.
- *Interval uncertainty:*
  - we measure the altitudes $y_1, \ldots, y_n$ at points $x_1 < \ldots < x_n$;
  - for measuring instruments, we usually have an upper bound Δ_i on the measurement error $\Delta y_i \stackrel{\rm def}{=} \widetilde{y}_i - y_i$.
  - thus, the only information that we have about the actual (unknown) value y_i is that $y_i \in \mathbf{y}_i = [\underline{y}_i, \overline{y}_i]$, where $\underline{y}_i \stackrel{\rm def}{=} \widetilde{y}_i - \Delta_i$ and $\overline{y}_i \stackrel{\rm def}{=} \widetilde{y}_i + \Delta_i$.


3
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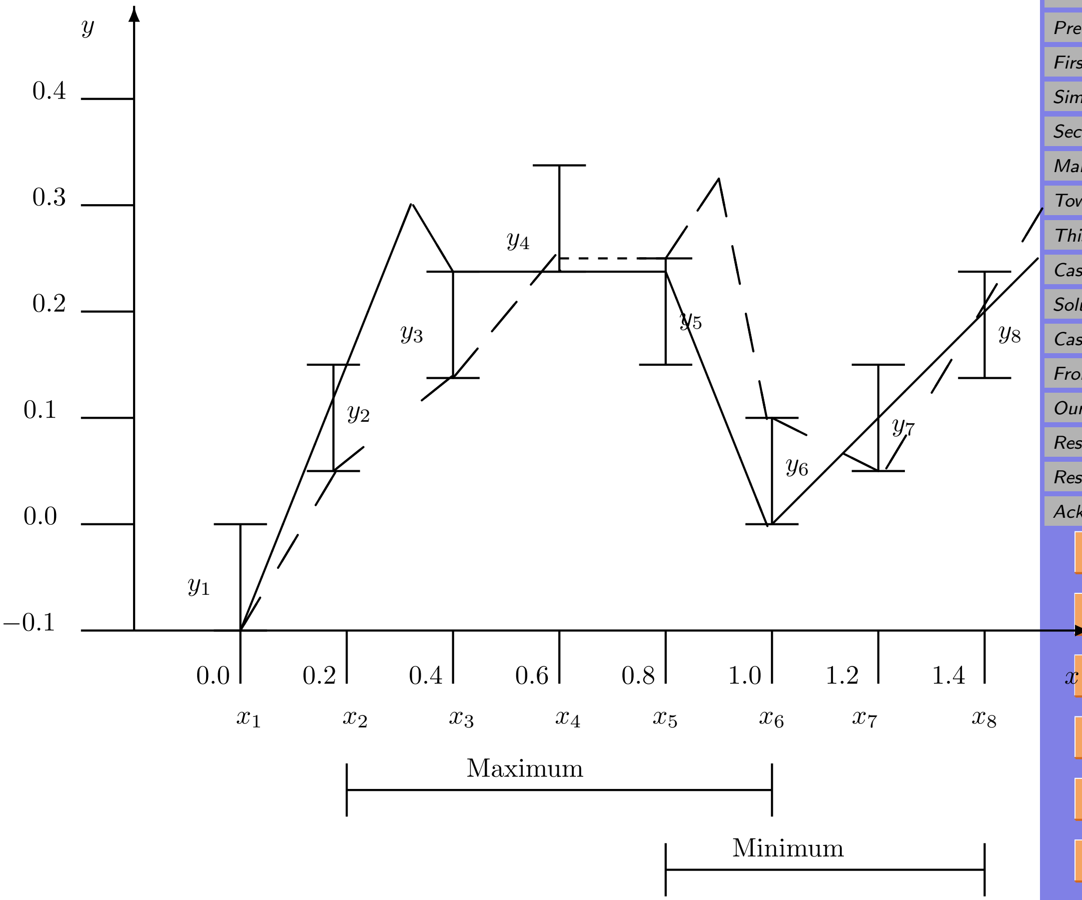

y
0.4
0.3
0.2
0.1
0.0
−0.1
y_1
y_2
y_3
y_4
y_5
y_6
y_7
y_8
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
x
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8
Maximum
Minimum

4. Toward a Formal Definition

- *Example:* we monitor the location y_i of a car on a highway at different moments x_i.
- We want to find out where the car was driving at the maximal allowed speed s.
- It is always possible that the car was going very fast when no one was looking (i.e., in between x_i and x_{i+1}).
- We are *not* interested in is not whether it is *possible* that somewhere, the slope is equal to s: it is always possible.
- We *are* interested in whether the data *imply* that somewhere, the slope was indeed equal to s, i.e.:

$$\forall f \in F \, \exists x \in [a,b] \, (f'(x) = s).$$

- *Comment:* this x may be different for different f.

5. Precise Formulation of the Problem

- An *interval function* F is a finite sequence of pairs $\langle x_i, \mathbf{y}_i \rangle$ $(i = 1, 2, \ldots, n)$, where:
  - $x_1 < x_2 < \ldots < x_n$ are real numbers, and
  - $\mathbf{y}_i$ are non-degenerate intervals.
- We say that $f \in F$ if $f(x)$ is continuously differentiable and $\forall i\,(f(x_i) \in \mathbf{y}_i)$.
- By a *derivative* $F'([a, b])$, we mean the intersection

$$F'([a, b]) \stackrel{\text{def}}{=} \bigcap_{f \in F} f'([a, b]),$$

  where $f'([a, b])$ is the range of f' over $[a, b]$.
- *Comment.* The notation $F'([a, b])$ looks like the range, but it is not: we can have $F'([a, b]) = \emptyset$.
- *Problem:* given F and $[a, b]$, compute $F'([a, b])$.

6. First Result: Checking Monotonicity

- *Definition.* We say that a function $f(x)$ is *strongly increasing* ($f \uparrow_s$) if $f'(x) > 0$ for all x.
- *Comment.* Every $f \uparrow_s$ is strictly increasing, but not vice versa: $f(x) = x^3$ is not $\uparrow_s$.
- *Result.* For every interval function F:

$$\exists f\,(f \in F \,\&\, f \uparrow_s) \leftrightarrow \forall i < j\,(\underline{y}_i < \overline{y}_j).$$

- *Proof.*

$\rightarrow$ If f is strongly increasing and $i < j$, then

$$\underline{y}_i \le f(x_i) < f(x_j) \le \overline{y}_j \text{ hence } \underline{y}_i < \overline{y}_j.$$

$\leftarrow$ Let $\Delta = \min(\overline{y}_i - \underline{y}_i) > 0$; take

$$y_i \stackrel{\text{def}}{=} \max(\underline{y}_1, \ldots, \underline{y}_i) + \frac{i}{2n} \cdot \Delta,$$

and choose a piece-wise linear function f for which $f(x_i) = y_i$.

7. Similar Results

- *Definition.* We say that a function $f(x)$ is *strongly decreasing* ($f \downarrow_s$) if $f'(x) < 0$ for all x.

- *Result.* For every interval function F:

$$\exists f\,(f \in F \,\&\, f \uparrow_s) \leftrightarrow \forall i < j\,(\overline{y}_i > \underline{y}_j).$$

- *Results about monotonicity on* $[a, b]$*:*
  - For every F and for every $[a, b]$, there exists $f \in F$ that is $\uparrow_s$ on $[a, b]$ if and only if:
  $$\forall i < j\,(x_i, x_j \in [a, b] \rightarrow \underline{y}_i < \overline{y}_j).$$
  - For every F and for every $[a, b]$, there exists $f \in F$ that is $\downarrow_s$ on $[a, b]$ if and only if:
  $$\forall i < j\,(x_i, x_j \in [a, b] \rightarrow \overline{y}_i > \underline{y}_j).$$

8. Second Result: Checking $0 \in F'([a,b])$

- *Result.* $0 \in F'([a,b]) \leftrightarrow$ neither of the following two conditions are satisfied:

$$\forall i < j\,(x_i, x_j \in [a,b] \to \underline{y}_i < \overline{y}_j);$$

$$\forall i < j\,(x_i, x_j \in [a,b] \to \overline{y}_i > \underline{y}_j).$$

- *Proof:* by reduction to a contradiction.

  $\rightarrow$ If one of these two conditions holds, then $f \uparrow_s$ or $f \downarrow_s$.

  * In both cases, $0 \notin F'([a,b])$ – contradiction.

  $\leftarrow$ If $0 \notin F'([a,b])$, i.e., $\forall x\,(f'(x) \neq 0)$, then $f'(x)$ is either always positive or always negative.

  * If $\forall x\,(f'(x) > 0)$, then the first condition holds – contradiction.
  * If $\forall x\,(f'(x) < 0)$, then the second condition holds – contradiction.

9. Main Result

- *Definition.* Let $F = \{\langle x_1, \mathbf{y}_1\rangle, \ldots, \langle x_n, \mathbf{y}_n\rangle\}$; define

$$F - v\cdot x \stackrel{\text{def}}{=} \{\langle x_1, \mathbf{y}_1 - v\cdot x_1\rangle, \ldots, \langle x_n, \mathbf{y}_n - v\cdot x_n\rangle\}.$$

- *Lemma.* $v \in F'([a,b]) \leftrightarrow 0 \in (F - v\cdot x)'([a,b])$.

- *Result.* Let i_0 and j_0 be the first and the last index of x_i inside $[a,b]$. Then $F'([a,b]) = [\underline{F}_{i_0,j_0}, \overline{F}_{i_0 j_0}]$, where

$$\underline{F}_{i_0 j_0} \stackrel{\text{def}}{=} \min_{i_0 \le i < j \le j_0} \overline{\Delta}_{ij}, \quad \overline{F}_{i_0 j_0} \stackrel{\text{def}}{=} \max_{i_0 \le i < j \le j_0} \underline{\Delta}_{ij},$$

$$\underline{\Delta}_{ij} \stackrel{\text{def}}{=} \frac{\underline{y}_i - \overline{y}_j}{x_j - x_i}, \quad \overline{\Delta}_{ij} \stackrel{\text{def}}{=} \frac{\overline{y}_i - \underline{y}_j}{x_j - x_i},$$

  where $[p,q] = \emptyset$ if $p > q$.

- *Comment.* The ratios $\underline{\Delta}_{ij}$ and $\overline{\Delta}_{ij}$ are finite differences – natural estimates for the derivatives.

- *Proof: idea.* $\neg\forall i < j\,(\underline{y}_i - v\cdot x_i < \overline{y}_j - v\cdot x_j) \leftrightarrow$ $\exists i < j\,(v\cdot(x_j - x_i) \ge \overline{y}_j - \underline{y}_i) \leftrightarrow \exists i < j\,(v \ge \overline{\Delta}_{ij}) \leftrightarrow v \ge \min\limits_{i<j} \overline{\Delta}_{ij} = \underline{F}_{i_0 j_0}$.

10. Towards a Faster Algorithm

- *Problem:* for each of $n \times n$ pairs $[a, b]$, we need to compute $O(n^2)$ values of $\underline{\Delta}_{ij}$ and $\overline{\Delta}_{ij}$ – too long.
- *Result.* There exists an algorithm that computes all possible values of $F'([a, b])$ in time $O(n^2)$:
  1. Compute $O(n^2)$ values $\underline{\Delta}_{ij}$ and $\overline{\Delta}_{ij}$ ($O(n^2)$ steps).
  2. For each i, compute $\overline{v}_{ij} \stackrel{\text{def}}{=} \max(\underline{\Delta}_{i,i+1}, \ldots, \underline{\Delta}_{ij})$ for $j = i+1, i+2, \ldots, n$ as follows:
     $$\overline{v}_{i,i+1} = \underline{\Delta}_{i,i+1}; \quad \overline{v}_{ij} = \max(\overline{v}_{i,j-1}, \underline{\Delta}_{ij}) \ (j > i + 1).$$
     For each i, we need $\le n$ steps – total of $O(n^2)$.
  3. $\forall j_0$, compute $\overline{F}_{i_0 j_0}$ for $i_0 = j_0 - 1, \ldots, 1$:
     $$\overline{F}_{j_0-1,j_0} = \underline{v}_{j_0-1,j_0}; \quad \overline{F}_{i_0,j_0} = \max(\overline{F}_{i_0+1,j_0}, \overline{v}_{i_0,j_0}).$$
     For each i, we need $\le n$ steps – total of $O(n^2)$.
- *Comment:* we need n^2 values, so $O(n^2)$ is optimal.

11. This Same Differential Formalism Also Serves an Alternative Definition of Zones

- *Problem:* a zone is sometimes defined by an interval $\mathbf{v} = [\underline{v}, \overline{v}]$ of possible values of $f'(x)$.
- *Result:* $\forall F\, \forall [a, b]$:

$$\forall f \in F\, (f'([a, b]) \cap \mathbf{v} \neq \emptyset) \leftrightarrow (\underline{F}_{i_0 j_0} \leq \overline{v}\, \&\, \overline{F}_{i_0 j_0} \geq \underline{v}).$$

- *Proof: idea.* We actually prove the equivalence of the two opposite statements:
  - there exists a function $f \in F$ for which

$$f'([a, b]) \cap \mathbf{v} = \emptyset;$$

  - $\underline{F}_{i_0 j_0} > \overline{v}$ or $\overline{F}_{i_0 j_0} < \underline{v}$.

12. Case Study: Dividing a Territory

- *Situations:* two sides compete for the same territory A.
- *Nash's solution:* the 1st side gets a set $X \subseteq A$ for which $u_1(X) \cdot u_2(A \setminus X) \to \max\limits_X$, where $u_1(X) = \int_X v_1(t)\,dt$ and $u_2(Y) = \int_Y v_2(t)\,dt$.
- *Objective:* solve this optimization problem simply, by equating some derivative to 0.
- *Similar case:* if an unknown f is a function, we can:
  - add small h to the value of $f(x)$ at x (and in a small vicinity of x) and
  - define $\dfrac{dU}{df}(x)$ as $\lim\limits_{h \to 0} \dfrac{U(f + \Delta f) - U(f)}{h}$.
- *Problem:* sets $\equiv$ characteristics functions (values $\in \{0, 1\}$), so we cannot add small h.

13. Solution

- *Definition:* the *derivative* of a set-function is

$$\frac{df}{dX}(t) = \lim_{H \in \mathcal{A}, H \to \{t\}, \mu_0(H) \neq 0} \frac{f(X \cup H) - f(X \setminus H)}{\mu_0(H)}$$

  (where the limit $H \to \{t\}$ is in the sense of Hausdorff metric).

- *Result:* If a continuously differentiable set function $f(X)$ attains its maximum at some set X, then:
  - $\frac{df}{dX}(t) = 0$ for all points $t \in \partial(\text{Int}(X)) \cap \partial(\text{Int}(CX))$.
  - for all $t \in \text{Int}(X)$, we have $\frac{df}{dX}(t) \geq 0$;
  - for all $t \in \text{Int}(CX)$, we have $\frac{df}{dX}(t) \leq 0$.

- *Territory division:* $\frac{df}{dX}(t) = v_1(t) \cdot u_2(A \setminus X) - v_2(t) \cdot u_1(X)$, so $X = \left\{t : \frac{v_1(t)}{v_2(t)} \geq \alpha\right\}$ for some α.

- *Excess mass:* the α-level contour $X_\alpha = \{t : \rho(t) \geq \alpha\}$ maximizes $\varepsilon(X) \stackrel{\text{def}}{=} \mu(X) - \alpha \cdot \mu_0(X)$.

14. Case Study: 5-D Space-Time

- *Success story:* in 1916, Einstein explained gravitation by combining space and time into a 4D space.
- *Natural question:* can we explain other physical fields by adding other physical dimensions?
- *Success:* in 1921, Kaluza and Klein showed that for the equations of general relativity theory in a 5D space:
  - the 4×4 components g_{ij} of the metric tensor still describe gravitation, while
  - the new components g^{5i} of the metric tensor satisfy Maxwell's equations.
- *Conclusion:* if we go to 5D space, we get a geometric interpretation of electrodynamics.

15. From Kaluza-Klein Model to Einstein-Bergman Interpretation

- *Problems with Kaluza-Klein model:*
  - in addition to physically meaningful g_{ij} and $A^i = g^{5i}$, we also get a physically meaningless field g_{55};
  - x^5 is also physically meaningless.
- *Physical solution* (Einstein and Bergmann 1938): the 5th dimension forms a tiny circle.
- *Modern physics:*
  - in 4-D space-time, equations lead to meaningless ∞ for physical quantities;
  - the smallest dimension where infinities disappear is 11 (super-string theory);
  - standard interpretation: all dimensions except the first four are tiny.

16. Our Objective

- *Starting point:* we have a cylinder $W = R^4 \times S^1$ instead of a linear space.
- *Objective:* to modify differential geometry by using W instead of R^4.
- *Problem:*
  - traditional differentiation uses the linear structure:
    - addition and
    - multiplication by a scalar;
  - we still have addition in W;
  - however, multiplication is not uniquely defined for angle-valued variables:
  - *example:* $0 \sim 2\pi$ while $0.6 \cdot 0 \not\sim 0.6 \cdot 2\pi$.

17. Resulting Formalism: Algebraic Part

- *Staring point:* the space W is not a vector space, only an Abelian group.
- *W-vectors* x^i are simply elements of W.
- *W-covectors* are *characters* on W, i.e., continuous additive maps $W \to S^1$.
- *Representation:* each covector $a \in V$ has the form $a(x) = \exp(\mathrm{i} \cdot a_i \cdot x^i)$ for some a_i.
- *Condition:* since $x^5 \sim x^5 + 2\pi$, we must have $a_5 \in Z$.
- *General case:* W-tensors are continuous additive mappings $W^k \times V^l \to S^1$.
- Here, $a = \exp(\mathrm{i} \cdot a^{j_1 \ldots j_l}_{i_1 \ldots i_k} \cdot x^{i_1} \cdot \ldots \cdot y^{i_k} \cdot z_{j_1} \ldots \cdot t_{j_l})$, where:
  - a_{55} is an integer;
  - $a_{ij5} = 0$ – adding 2π to z^5 should not change $a(x, y, z)$, so $a_{ij5} \cdot x^i \cdot y^j \cdot 2\pi \equiv 0$ hence $a_{ij5} = 0$.

18. Resulting Formalism: Differential Part

- *Main idea:* a W-tensor field $a^{j_1\ldots j_l}_{i_1\ldots i_k}(x^b)$ is W-differentiable if its derivatives

$$a^{j_1\ldots j_l}_{i_1\ldots i_k,b} \stackrel{\text{def}}{=} \frac{\partial a^{j_1\ldots j_l}_{i_1\ldots i_k}}{\partial x^b}$$

  also form a W-tensor.

- *First corollary:*
  - the field $g_{55}(x^b)$ is differentiable;
  - it only takes integer values;
  - so, it is a constant – not a physical field.

- *Second corollary:* for g_{ij}, we get $g_{ij,5} = \dfrac{\partial g_{ij}}{\partial x^5} = 0$.

- *Conclusion:* we explained both requirements by the fact that we have a cylinder instead of a linear space.

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