

Dealing with Uncertainties in Computing: from Probabilistic and Interval Uncertainty to Combination of Different Approaches, with Application to Geoinformatics, Bioinformatics, and Engineering

Vladik Kreinovich

Department of Computer Science,
University of Texas at El Paso,
El Paso, TX 79968, USA
vladik@utep.edu
<http://www.cs.utep.edu/vladik>

Interval computations website:
<http://www.cs.utep.edu/interval-comp>

General Problem of...

Interval Arithmetic:...

Case Study: Chip Design

Combining Interval...

Case Study:...

Case Study: Detecting...

Acknowledgments

Fuzzy Computations:...

Title Page

◀

▶

◀

▶

Page 1 of 53

Go Back

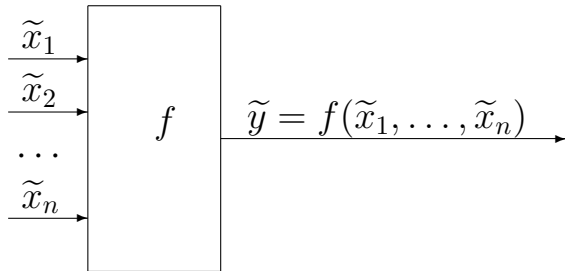
Full Screen

Close

Quit

1. General Problem of Data Processing under Uncertainty

- *Indirect measurements*: way to measure y that are difficult (or even impossible) to measure directly.
- *Idea*: $y = f(x_1, \dots, x_n)$



- *Problem*: measurements are never 100% accurate: $\tilde{x}_i \neq x_i$ ($\Delta x_i \neq 0$) hence

$$\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n) \neq y = f(x_1, \dots, x_n).$$

What are bounds on $\Delta y \stackrel{\text{def}}{=} \tilde{y} - y$?

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 2 of 53

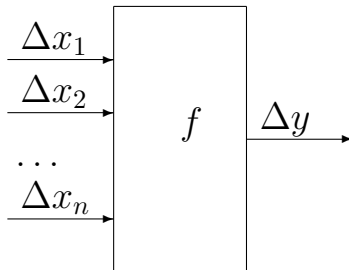
Go Back

Full Screen

Close

Quit

2. Probabilistic and Interval Uncertainty



- *Traditional approach:* we know probability distribution for Δx_i (usually Gaussian).
- *Where it comes from:* calibration using standard MI.
- *Problem:* calibration is not possible in:
 - fundamental science
 - manufacturing
- *Solution:* we know upper bounds Δ_i on $|\Delta x_i|$ hence

$$x_i \in [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i].$$

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 3 of 53

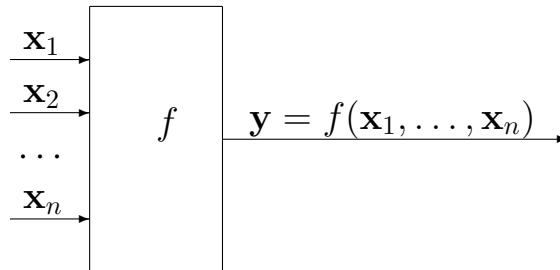
Go Back

Full Screen

Close

Quit

3. Interval Computations: A Problem



- *Given:* an algorithm $y = f(x_1, \dots, x_n)$ and n intervals $\mathbf{x}_i = [\underline{x}_i, \bar{x}_i]$.
- *Compute:* the corresponding range of y :
$$[\underline{y}, \bar{y}] = \{f(x_1, \dots, x_n) \mid x_1 \in [\underline{x}_1, \bar{x}_1], \dots, x_n \in [\underline{x}_n, \bar{x}_n]\}.$$
- *Fact:* NP-hard even for quadratic f .
- *Challenge:* when are feasible algorithms possible?
- *Challenge:* when computing $\mathbf{y} = [\underline{y}, \bar{y}]$ is not feasible, find a good approximation $\mathbf{Y} \supseteq \mathbf{y}$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 4 of 53

Go Back

Full Screen

Close

Quit

4. Interval Computations: A Brief History

- *Origins*: Archimedes (Ancient Greece)
- *Modern pioneers*: Warmus (Poland), Sunaga (Japan), Moore (USA), 1956–59
- *First boom*: early 1960s.
- *First challenge*: taking interval uncertainty into account when planning spaceflights to the Moon.
- *Current applications* (sample):
 - design of elementary particle colliders: Berz, Kyoko (USA)
 - will a comet hit the Earth: Berz, Moore (USA)
 - robotics: Jaulin (France), Neumaier (Austria)
 - chemical engineering: Stadtherr (USA)

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 5 of 53

Go Back

Full Screen

Close

Quit

5. Alternative Approach: Maximum Entropy

- *Situation*: in many practical applications, it is very difficult to come up with the probabilities.
- *Traditional engineering approach*: use probabilistic techniques.
- *Problem*: many different probability distributions are consistent with the same observations.
- *Solution*: select one of these distributions – e.g., the one with the largest entropy.
- *Example – single variable*: if all we know is that $x \in [\underline{x}, \bar{x}]$, then MaxEnt leads to a uniform distribution on $[\underline{x}, \bar{x}]$.
- *Example – multiple variables*: different variables are independently distributed.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 6 of 53

Go Back

Full Screen

Close

Quit

6. Limitations of Maximum Entropy Approach

- *Example:* simplest algorithm $y = x_1 + \dots + x_n$.
- *Measurement errors:* $\Delta x_i \in [-\Delta, \Delta]$.
- *Analysis:* $\Delta y = \Delta x_1 + \dots + \Delta x_n$.
- *Worst case situation:* $\Delta y = n \cdot \Delta$.
- *Maximum Entropy approach:* due to Central Limit Theorem, Δy is $\approx$ normal, with $\sigma = \Delta \cdot \frac{\sqrt{n}}{\sqrt{3}}$.
- *Why this may be inadequate:* we get $\Delta \sim \sqrt{n}$, but due to correlation, it is possible that $\Delta = n \cdot \Delta \sim n \gg \sqrt{n}$.
- *Conclusion:* using a single distribution can be very misleading, especially if we want guaranteed results.
- *Examples:* high-risk application areas such as space exploration or nuclear engineering.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 7 of 53

Go Back

Full Screen

Close

Quit

7. Interval Arithmetic: Foundations of Interval Techniques

- *Problem:* compute the range

$$[\underline{y}, \bar{y}] = \{f(x_1, \dots, x_n) \mid x_1 \in [\underline{x}_1, \bar{x}_1], \dots, x_n \in [\underline{x}_n, \bar{x}_n]\}.$$

- *Interval arithmetic:* for arithmetic operations $f(x_1, x_2)$ (and for elementary functions), we have explicit formulas for the range.
- *Examples:* when $x_1 \in \mathbf{x}_1 = [\underline{x}_1, \bar{x}_1]$ and $x_2 \in \mathbf{x}_2 = [\underline{x}_2, \bar{x}_2]$, then:
 - The range $\mathbf{x}_1 + \mathbf{x}_2$ for $x_1 + x_2$ is $[\underline{x}_1 + \underline{x}_2, \bar{x}_1 + \bar{x}_2]$.
 - The range $\mathbf{x}_1 - \mathbf{x}_2$ for $x_1 - x_2$ is $[\underline{x}_1 - \bar{x}_2, \bar{x}_1 - \underline{x}_2]$.
 - The range $\mathbf{x}_1 \cdot \mathbf{x}_2$ for $x_1 \cdot x_2$ is $[\underline{y}, \bar{y}]$, where
$$\underline{y} = \min(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \bar{x}_2, \bar{x}_1 \cdot \underline{x}_2, \bar{x}_1 \cdot \bar{x}_2);$$
$$\bar{y} = \max(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \bar{x}_2, \bar{x}_1 \cdot \underline{x}_2, \bar{x}_1 \cdot \bar{x}_2).$$
- The range $1/\mathbf{x}_1$ for $1/x_1$ is $[1/\bar{x}_1, 1/\underline{x}_1]$ (if $0 \notin \mathbf{x}_1$).

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 8 of 53

Go Back

Full Screen

Close

Quit

8. Straightforward Interval Computations: Example

- *Example:* $f(x) = (x - 2) \cdot (x + 2)$, $x \in [1, 2]$.
- How will the computer compute it?
 - $r_1 := x - 2$;
 - $r_2 := x + 2$;
 - $r_3 := r_1 \cdot r_2$.
- *Main idea:* perform the same operations, but with *intervals* instead of *numbers*:
 - $\mathbf{r}_1 := [1, 2] - [2, 2] = [-1, 0]$;
 - $\mathbf{r}_2 := [1, 2] + [2, 2] = [3, 4]$;
 - $\mathbf{r}_3 := [-1, 0] \cdot [3, 4] = [-4, 0]$.
- *Actual range:* $f(\mathbf{x}) = [-3, 0]$.
- *Comment:* this is just a toy example, there are more efficient ways of computing an enclosure $\mathbf{Y} \supseteq \mathbf{y}$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 9 of 53

Go Back

Full Screen

Close

Quit

9. First Idea: Use of Monotonicity

- *Reminder:* for arithmetic, we had exact ranges.
- *Reason:* $+$, $-$, $\cdot$ are monotonic in each variable.
- *How monotonicity helps:* if $f(x_1, \dots, x_n)$ is (non-strictly) increasing ($f \uparrow$) in each x_i , then

$$f(\mathbf{x}_1, \dots, \mathbf{x}_n) = [f(\underline{x}_1, \dots, \underline{x}_n), f(\bar{x}_1, \dots, \bar{x}_n)].$$

- *Similarly:* if $f \uparrow$ for some x_i and $f \downarrow$ for other x_j .
- *Fact:* $f \uparrow$ in x_i if $\frac{\partial f}{\partial x_i} \geq 0$.
- *Checking monotonicity:* check that the range $[\underline{r}_i, \bar{r}_i]$ of $\frac{\partial f}{\partial x_i}$ on $\mathbf{x}_i$ has $\underline{r}_i \geq 0$.
- *Differentiation:* by Automatic Differentiation (AD) tools.
- *Estimating ranges of $\frac{\partial f}{\partial x_i}$:* straightforward interval comp.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 10 of 53

Go Back

Full Screen

Close

Quit

10. Monotonicity: Example

- *Idea:* if the range $[\underline{r}_i, \bar{r}_i]$ of each $\frac{\partial f}{\partial x_i}$ on $\mathbf{x}_i$ has $\underline{r}_i \geq 0$, then

$$f(\mathbf{x}_1, \dots, \mathbf{x}_n) = [f(\underline{x}_1, \dots, \underline{x}_n), f(\bar{x}_1, \dots, \bar{x}_n)].$$

- *Example:* $f(x) = (x - 2) \cdot (x + 2)$, $\mathbf{x} = [1, 2]$.
- *Case $n = 1$:* if the range $[\underline{r}, \bar{r}]$ of $\frac{df}{dx}$ on $\mathbf{x}$ has $\underline{r} \geq 0$, then

$$f(\mathbf{x}) = [f(\underline{x}), f(\bar{x})].$$

- *AD:* $\frac{df}{dx} = 1 \cdot (x + 2) + (x - 2) \cdot 1 = 2x$.
- *Checking:* $[\underline{r}, \bar{r}] = [2, 4]$, with $2 \geq 0$.
- *Result:* $f([1, 2]) = [f(1), f(2)] = [-3, 0]$.
- *Comparison:* this is the exact range.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 11 of 53

Go Back

Full Screen

Close

Quit

11. Non-Monotonic Example

- *Example:* $f(x) = x \cdot (1 - x)$, $x \in [0, 1]$.
- How will the computer compute it?
 - $r_1 := 1 - x$;
 - $r_2 := x \cdot r_1$.
- *Straightforward interval computations:*
 - $\mathbf{r}_1 := [1, 1] - [0, 1] = [0, 1]$;
 - $\mathbf{r}_2 := [0, 1] \cdot [0, 1] = [0, 1]$.
- *Actual range:* min, max of f at $\underline{x}$, $\bar{x}$, or when $\frac{df}{dx} = 0$.
- Here, $\frac{df}{dx} = 1 - 2x = 0$ for $x = 0.5$, so
 - compute $f(0) = 0$, $f(0.5) = 0.25$, and $f(1) = 0$.
 - $\underline{y} = \min(0, 0.25, 0) = 0$, $\bar{y} = \max(0, 0.25, 0) = 0.25$.
- *Resulting range:* $f(\mathbf{x}) = [0, 0.25]$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 12 of 53

Go Back

Full Screen

Close

Quit

12. Second Idea: Centered Form

- *Main idea:* Intermediate Value Theorem

$$f(x_1, \dots, x_n) = f(\tilde{x}_1, \dots, \tilde{x}_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\chi) \cdot (x_i - \tilde{x}_i)$$

for some $\chi_i \in \mathbf{x}_i$.

- *Corollary:* $f(x_1, \dots, x_n) \in \mathbf{Y}$, where

$$\mathbf{Y} = \tilde{y} + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}_1, \dots, \mathbf{x}_n) \cdot [-\Delta_i, \Delta_i].$$

- *Differentiation:* by Automatic Differentiation (AD) tools.
- *Estimating the ranges of derivatives:*
 - if appropriate, by monotonicity, or
 - by straightforward interval computations, or
 - by centered form (more time but more accurate).

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 13 of 53

Go Back

Full Screen

Close

Quit

13. Centered Form: Example

- *General formula:*

$$\mathbf{Y} = f(\tilde{x}_1, \dots, \tilde{x}_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\mathbf{x}_1, \dots, \mathbf{x}_n) \cdot [-\Delta_i, \Delta_i].$$

- *Example:* $f(x) = x \cdot (1 - x)$, $\mathbf{x} = [0, 1]$.
- Here, $\mathbf{x} = [\tilde{x} - \Delta, \tilde{x} + \Delta]$, with $\tilde{x} = 0.5$ and $\Delta = 0.5$.
- *Case $n = 1$:* $\mathbf{Y} = f(\tilde{x}) + \frac{df}{dx}(\mathbf{x}) \cdot [-\Delta, \Delta]$.
- *AD:* $\frac{df}{dx} = 1 \cdot (1 - x) + x \cdot (-1) = 1 - 2x$.
- *Estimation:* we have $\frac{df}{dx}(\mathbf{x}) = 1 - 2 \cdot [0, 1] = [-1, 1]$.
- *Result:* $\mathbf{Y} = 0.5 \cdot (1 - 0.5) + [-1, 1] \cdot [-0.5, 0.5] = 0.25 + [-0.5, 0.5] = [-0.25, 0.75]$.
- *Comparison:* actual range $[0, 0.25]$, straightforward $[0, 1]$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 14 of 53

Go Back

Full Screen

Close

Quit

14. Third Idea: Bisection

- *Known:* accuracy $O(\Delta_i^2)$ of first order formula

$$f(x_1, \dots, x_n) = f(\tilde{x}_1, \dots, \tilde{x}_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\chi) \cdot (x_i - \tilde{x}_i).$$

- *Idea:* if the intervals are too wide, we:
 - split one of them in half ($\Delta_i^2 \rightarrow \Delta_i^2/4$); and
 - take the union of the resulting ranges.
- *Example:* $f(x) = x \cdot (1 - x)$, where $x \in \mathbf{x} = [0, 1]$.
- *Split:* take $\mathbf{x}' = [0, 0.5]$ and $\mathbf{x}'' = [0.5, 1]$.
- *1st range:* $1 - 2 \cdot \mathbf{x} = 1 - 2 \cdot [0, 0.5] = [0, 1]$, so $f \uparrow$ and $f(\mathbf{x}') = [f(0), f(0.5)] = [0, 0.25]$.
- *2nd range:* $1 - 2 \cdot \mathbf{x} = 1 - 2 \cdot [0.5, 1] = [-1, 0]$, so $f \downarrow$ and $f(\mathbf{x}'') = [f(1), f(0.5)] = [0, 0.25]$.
- *Result:* $f(\mathbf{x}') \cup f(\mathbf{x}'') = [0, 0.25]$ – exact.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 15 of 53

Go Back

Full Screen

Close

Quit

15. Alternative Approach: Affine Arithmetic

- *So far:* we compute the range of $x \cdot (1 - x)$ by multiplying ranges of x and $1 - x$.
- *We ignore:* that both factors depend on x and are, thus, dependent.
- *Idea:* for each intermediate result a , keep an explicit dependence on $\Delta x_i = \tilde{x}_i - x_i$ (at least its linear terms).
- *Implementation:*

$$a = a_0 + \sum_{i=1}^n a_i \cdot \Delta x_i + [\underline{a}, \bar{a}].$$

- *We start:* with $x_i = \tilde{x}_i - \Delta x_i$, i.e.,
 $\tilde{x}_i + 0 \cdot \Delta x_1 + \dots + 0 \cdot \Delta x_{i-1} + (-1) \cdot \Delta x_i + 0 \cdot \Delta x_{i+1} + \dots + 0 \cdot \Delta x_n + [0, 0]$.
- *Description:* $a_0 = \tilde{x}_i$, $a_i = -1$, $a_j = 0$ for $j \neq i$, and $[\underline{a}, \bar{a}] = [0, 0]$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 16 of 53

Go Back

Full Screen

Close

Quit

16. Affine Arithmetic: Operations

- *Representation:* $a = a_0 + \sum_{i=1}^n a_i \cdot \Delta x_i + [\underline{a}, \bar{a}]$.
- *Input:* $a = a_0 + \sum_{i=1}^n a_i \cdot \Delta x_i + \mathbf{a}$ and $b = b_0 + \sum_{i=1}^n b_i \cdot \Delta x_i + \mathbf{b}$.
- *Operations:* $c = a \otimes b$.
- *Addition:* $c_0 = a_0 + b_0$, $c_i = a_i + b_i$, $\mathbf{c} = \mathbf{a} + \mathbf{b}$.
- *Subtraction:* $c_0 = a_0 - b_0$, $c_i = a_i - b_i$, $\mathbf{c} = \mathbf{a} - \mathbf{b}$.
- *Multiplication:* $c_0 = a_0 \cdot b_0$, $c_i = a_0 \cdot b_i + b_0 \cdot a_i$,
 $\mathbf{c} = a_0 \cdot \mathbf{b} + b_0 \cdot \mathbf{a} + \sum_{i \neq j} a_i \cdot b_j \cdot [-\Delta_i, \Delta_i] \cdot [-\Delta_j, \Delta_j] +$
 $\sum_i a_i \cdot b_i \cdot [-\Delta_i, \Delta_i]^2 +$
 $\left(\sum_i a_i \cdot [-\Delta_i, \Delta_i] \right) \cdot \mathbf{b} + \left(\sum_i b_i \cdot [-\Delta_i, \Delta_i] \right) \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b}.$

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 17 of 53

Go Back

Full Screen

Close

Quit

17. Affine Arithmetic: Example

- *Example:* $f(x) = x \cdot (1 - x)$, $x \in [0, 1]$.
- Here, $n = 1$, $\tilde{x} = 0.5$, and $\Delta = 0.5$.
- How will the computer compute it?
 - $r_1 := 1 - x$;
 - $r_2 := x \cdot r_1$.
- *Affine arithmetic:* we start with $x = 0.5 - \Delta x + [0, 0]$;
 - $\mathbf{r}_1 := 1 - (0.5 - \Delta x) = 0.5 + \Delta x$;
 - $\mathbf{r}_2 := (0.5 - \Delta x) \cdot (0.5 + \Delta x)$, i.e.,
$$\mathbf{r}_2 = 0.25 + 0 \cdot \Delta x - [-\Delta, \Delta]^2 = 0.25 + [-\Delta^2, 0].$$
- *Resulting range:* $\mathbf{y} = 0.25 + [-0.25, 0] = [0, 0.25]$.
- *Comparison:* this is the exact range.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 18 of 53

Go Back

Full Screen

Close

Quit

18. Affine Arithmetic: Towards More Accurate Estimates

- *In our simple example:* we got the exact range.
 - *In general:* range estimation is NP-hard.
 - *Meaning:* a feasible (polynomial-time) algorithm will sometimes lead to excess width: $\mathbf{Y} \supset \mathbf{y}$.
 - *Conclusion:* affine arithmetic may lead to excess width.
 - *Question:* how to get more accurate estimates?
 - *First idea:* bisection.
 - *Second idea* (Taylor arithmetic):
 - *affine arithmetic:* $a = a_0 + \sum a_i \cdot \Delta x_i + \mathbf{a}$;
 - *meaning:* we keep linear terms in Δx_i ;
 - *idea:* keep, e.g., quadratic terms
- $$a = a_0 + \sum a_i \cdot \Delta x_i + \sum a_{ij} \cdot \Delta x_i \cdot \Delta x_j + \mathbf{a}.$$

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 19 of 53

Go Back

Full Screen

Close

Quit

19. Interval Computations vs. Affine Arithmetic: Comparative Analysis

- *Objective:* we want a method that computes a reasonable estimate for the range in reasonable time.
- *Conclusion – how to compare different methods:*
 - how accurate are the estimates, and
 - how fast we can compute them.
- *Accuracy:* affine arithmetic leads to more accurate ranges.
- *Computation time:*
 - *Interval arithmetic:* for each intermediate result a , we compute two values: endpoints $\underline{a}$ and $\bar{a}$ of $[\underline{a}, \bar{a}]$.
 - *Affine arithmetic:* for each a , we compute $n + 3$ values:

$$a_0 \quad a_1, \dots, a_n \quad \underline{a}, \bar{a}.$$

- *Conclusion:* affine arithmetic is $\sim n$ times slower.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 20 of 53

Go Back

Full Screen

Close

Quit

20. Solving Systems of Equations: Extending Known Algorithms to Situations with Interval Uncertainty

- *We have:* a system of equations $g_i(y_1, \dots, y_n) = a_i$ with unknowns y_i ;
- *We know:* a_i with interval uncertainty: $a_i \in [\underline{a}_i, \bar{a}_i]$;
- *We want:* to find the corresponding ranges of y_j .
- *First case:* for exactly known a_i , we have an algorithm $y_j = f_j(a_1, \dots, a_n)$ for solving the system.
- *Example:* system of linear equations.
- *Solution:* apply interval computations techniques to find the range $f_j([\underline{a}_1, \bar{a}_1], \dots, [\underline{a}_n, \bar{a}_n])$.
- *Better solution:* for specific equations, we often already know which ideas work best.
- *Example:* linear equations $Ay = b$; y is monotonic in b .

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 21 of 53

Go Back

Full Screen

Close

Quit

21. Solving Systems of Equations When No Algorithm Is Known

- *Idea:*
 - parse each equation into elementary constraints, and
 - use interval computations to improve original ranges until we get a narrow range (= solution).
- *First example:* $x - x^2 = 0.5$, $x \in [0, 1]$ (no solution).
- *Parsing:* $r_1 = x^2$, $0.5 (= r_2) = x - r_1$.
- *Rules:* from $r_1 = x^2$, we extract two rules:

$$(1) x \rightarrow r_1 = x^2; \quad (2) r_1 \rightarrow x = \sqrt{r_1};$$

from $0.5 = x - r_1$, we extract two more rules:

$$(3) x \rightarrow r_1 = x - 0.5; \quad (4) r_1 \rightarrow x = r_1 + 0.5.$$

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 22 of 53

Go Back

Full Screen

Close

Quit

22. Solving Systems of Equations When No Algorithm Is Known: Example

- (1) $r = x^2$; (2) $x = \sqrt{r}$; (3) $r = x - 0.5$; (4) $x = r + 0.5$.

- We start with: $\mathbf{x} = [0, 1]$, $\mathbf{r} = (-\infty, \infty)$.

(1) $\mathbf{r} = [0, 1]^2 = [0, 1]$, so $\mathbf{r}_{\text{new}} = (-\infty, \infty) \cap [0, 1] = [0, 1]$.

(2) $\mathbf{x}_{\text{new}} = \sqrt{[0, 1]} \cap [0, 1] = [0, 1]$ – no change.

(3) $\mathbf{r}_{\text{new}} = ([0, 1] - 0.5) \cap [0, 1] = [-0.5, 0.5] \cap [0, 1] = [0, 0.5]$.

(4) $\mathbf{x}_{\text{new}} = ([0, 0.5] + 0.5) \cap [0, 1] = [0.5, 1] \cap [0, 1] = [0.5, 1]$.

(1) $\mathbf{r}_{\text{new}} = [0.5, 1]^2 \cap [0, 0.5] = [0.25, 0.5]$.

(2) $\mathbf{x}_{\text{new}} = \sqrt{[0.25, 0.5]} \cap [0.5, 1] = [0.5, 0.71]$;
round $\underline{a}$ down $\downarrow$ and $\bar{a}$ up $\uparrow$, to guarantee enclosure.

(3) $\mathbf{r}_{\text{new}} = ([0.5, 0.71] - 0.5) \cap [0.25, 0.5] = [0.0, 0.21] \cap [0.25, 0.5]$,
i.e., $\mathbf{r}_{\text{new}} = \emptyset$.

- *Conclusion:* the original equation has no solutions.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 23 of 53

Go Back

Full Screen

Close

Quit

23. Solving Systems of Equations: Second Example

- *Example:* $x - x^2 = 0$, $x \in [0, 1]$.
- *Parsing:* $r_1 = x^2$, $0 (= r_2) = x - r_1$.
- *Rules:* (1) $r = x^2$; (2) $x = \sqrt{r}$; (3) $r = x$; (4) $x = r$.
- *We start with:* $\mathbf{x} = [0, 1]$, $\mathbf{r} = (-\infty, \infty)$.
- *Problem:* after Rule 1, we're stuck with $\mathbf{x} = \mathbf{r} = [0, 1]$.
- *Solution:* bisect $\mathbf{x} = [0, 1]$ into $[0, 0.5]$ and $[0.5, 1]$.
- *For 1st subinterval:*
 - Rule 1 leads to $\mathbf{r}_{\text{new}} = [0, 0.5]^2 \cap [0, 0.5] = [0, 0.25]$;
 - Rule 4 leads to $\mathbf{x}_{\text{new}} = [0, 0.25]$;
 - Rule 1 leads to $\mathbf{r}_{\text{new}} = [0, 0.25]^2 = [0, 0.0625]$;
 - Rule 4 leads to $\mathbf{x}_{\text{new}} = [0, 0.0625]$; etc.
 - we converge to $x = 0$.
- *For 2nd subinterval:* we converge to $x = 1$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 24 of 53

Go Back

Full Screen

Close

Quit

24. Optimization: Extending Known Algorithms to Situations with Interval Uncertainty

- *Problem:* find $y_1, \dots, y_m$ for which

$$g(y_1, \dots, y_m, a_1, \dots, a_m) \rightarrow \max.$$

- *We know:* a_i with interval uncertainty: $a_i \in [\underline{a}_i, \bar{a}_i]$;
- *We want:* to find the corresponding ranges of y_j .
- *First case:* for exactly known a_i , we have an algorithm $y_j = f_j(a_1, \dots, a_n)$ for solving the optimization problem.
- *Example:* quadratic objective function g .
- *Solution:* apply interval computations techniques to find the range $f_j([\underline{a}_1, \bar{a}_1], \dots, [\underline{a}_n, \bar{a}_n])$.
- *Better solution:* for specific f , we often already know which ideas work best.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 25 of 53

Go Back

Full Screen

Close

Quit

25. Optimization When No Algorithm Is Known

- *Idea:* divide the original box $\mathbf{x}$ into subboxes $\mathbf{b}$.
- If $\max_{x \in \mathbf{b}} g(x) < g(x')$ for a known x' , dismiss $\mathbf{b}$.
- *Example:* $g(x) = x \cdot (1 - x)$, $\mathbf{x} = [0, 1]$.
- Divide into 10 (?) subboxes $\mathbf{b} = [0, 0.1], [0.1, 0.2], \dots$
- Find $g(\tilde{\mathbf{b}})$ for each $\mathbf{b}$; the largest is $0.45 \cdot 0.55 = 0.2475$.
- Compute $G(\mathbf{b}) = g(\tilde{\mathbf{b}}) + (1 - 2 \cdot \mathbf{b}) \cdot [-\Delta, \Delta]$.
- Dismiss subboxes for which $\bar{Y} < 0.2475$.
- *Example:* for $[0.2, 0.3]$, we have
$$0.25 \cdot (1 - 0.25) + (1 - 2 \cdot [0.2, 0.3]) \cdot [-0.05, 0.05].$$
- Here $\bar{Y} = 0.2175 < 0.2475$, so we dismiss $[0.2, 0.3]$.
- *Result:* keep only boxes $\subseteq [0.3, 0.7]$.
- *Further subdivision:* get us closer and closer to $x = 0.5$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 26 of 53

Go Back

Full Screen

Close

Quit

26. Case Study: Chip Design

- *Chip design*: one of the main objectives is to decrease the clock cycle.
- *Current approach*: uses worst-case (interval) techniques.
- *Problem*: the probability of the worst-case values is usually very small.
- *Result*: estimates are over-conservative – unnecessary over-design and under-performance of circuits.
- *Difficulty*: we only have *partial* information about the corresponding probability distributions.
- *Objective*: produce estimates valid for all distributions which are consistent with this information.
- *What we do*: provide such estimates for the clock time.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page



Page 27 of 53

Go Back

Full Screen

Close

Quit

27. Estimating Clock Cycle: a Practical Problem

- *Objective:* estimate the clock cycle on the design stage.
- The clock cycle of a chip is constrained by the maximum path delay over all the circuit paths

$$D \stackrel{\text{def}}{=} \max(D_1, \dots, D_N).$$

- The path delay D_i along the i -th path is the sum of the delays corresponding to the gates and wires along this path.
- Each of these delays, in turn, depends on several factors such as:
 - the variation caused by the current design practices,
 - environmental design characteristics (e.g., variations in temperature and in supply voltage), etc.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 28 of 53

Go Back

Full Screen

Close

Quit

28. Traditional (Interval) Approach to Estimating the Clock Cycle

- *Traditional approach:* assume that each factor takes the worst possible value.
- *Result:* time delay when all the factors are at their worst.
- *Problem:*
 - different factors are usually independent;
 - combination of worst cases is improbable.
- *Computational result:* current estimates are 30% above the observed clock time.
- *Practical result:* the clock time is set too high – chips are over-designed and under-performing.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 29 of 53

Go Back

Full Screen

Close

Quit

29. Robust Statistical Methods Are Needed

- *Ideal case*: we know probability distributions.
- *Solution*: Monte-Carlo simulations.
- *In practice*: we only have *partial* information about the distributions of some of the parameters; usually:
 - the mean, and
 - some characteristic of the deviation from the mean
 - e.g., the interval that is guaranteed to contain possible values of this parameter.
- *Possible approach*: Monte-Carlo with several possible distributions.
- *Problem*: no guarantee that the result is a valid bound for all possible distributions.
- *Objective*: provide *robust* bounds, i.e., bounds that work for all possible distributions.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 30 of 53

Go Back

Full Screen

Close

Quit

30. Towards a Mathematical Formulation of the Problem

- *General case:* each gate delay d depends on the difference $x_1, \dots, x_n$ between the actual and the nominal values of the parameters.
- *Main assumption:* these differences are usually small.
- Each path delay D_i is the sum of gate delays.
- *Conclusion:* D_i is a linear function: $D_i = a_i + \sum_{j=1}^n a_{ij} \cdot x_j$
for some a_i and a_{ij} .
- The desired maximum delay $D = \max_i D_i$ has the form

$$D = F(x_1, \dots, x_n) \stackrel{\text{def}}{=} \max_i \left(a_i + \sum_{j=1}^n a_{ij} \cdot x_j \right).$$

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 31 of 53

Go Back

Full Screen

Close

Quit

31. Towards a Mathematical Formulation of the Problem (cont-d)

- *Known*: maxima of linear function are exactly convex functions:

$$F(\alpha \cdot x + (1 - \alpha) \cdot y) \leq \alpha \cdot F(x) + (1 - \alpha) \cdot F(y)$$

for all x, y and for all $\alpha \in [0, 1]$;

- *We know*: factors x_i are independent;
 - we know distribution of some of the factors;
 - for others, we know ranges $[x_j, \bar{x}_j]$ and means E_j .
- *Given*: a convex function $F \geq 0$ and a number $\varepsilon > 0$.
- *Objective*: find the smallest y_0 s.t. for all possible distributions, we have $y \leq y_0$ with the probability $\geq 1 - \varepsilon$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page



Page 32 of 53

Go Back

Full Screen

Close

Quit

32. Additional Property: Dependency is Non-Degenerate

- *Fact*: sometimes, we learn additional information about one of the factors x_j .
- *Example*: we learn that x_j actually belongs to a proper subinterval of the original interval $[\underline{x}_j, \bar{x}_j]$.
- *Consequence*: the class $\mathcal{P}$ of possible distributions is replaced with $\mathcal{P}' \subset \mathcal{P}$.
- *Result*: the new value y'_0 can only decrease: $y'_0 \leq y_0$.
- *Fact*: if x_j is irrelevant for y , then $y'_0 = y_0$.
- *Assumption*: irrelevant variables been weeded out.
- *Formalization*: if we narrow down one of the intervals $[\underline{x}_j, \bar{x}_j]$, the resulting value y_0 decreases: $y'_0 < y_0$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 33 of 53

Go Back

Full Screen

Close

Quit

33. Formulation of the Problem

GIVEN: • $n, k \leq n, \varepsilon > 0$;

• a convex function $y = F(x_1, \dots, x_n) \geq 0$;

• $n - k$ cdfs $F_j(x)$, $k + 1 \leq j \leq n$;

• intervals $\mathbf{x}_1, \dots, \mathbf{x}_k$, values $E_1, \dots, E_k$,

TAKE: all joint probability distributions on R^n for which:

• all x_i are independent,

• $x_j \in \mathbf{x}_j$, $E[x_j] = E_j$ for $j \leq k$, and

• x_j have distribution $F_j(x)$ for $j > k$.

FIND: the smallest y_0 s.t. for all such distributions,

$F(x_1, \dots, x_n) \leq y_0$ with probability $\geq 1 - \varepsilon$.

WHEN: the problem is *non-degenerate* – if we narrow down one of the intervals $\mathbf{x}_j$, y_0 decreases.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 34 of 53

Go Back

Full Screen

Close

Quit

34. Main Result and How We Can Use It

- *Result:* y_0 is attained when for each j from 1 to k ,

- $x_j = \underline{x}_j$ with probability $\underline{p}_j \stackrel{\text{def}}{=} \frac{\bar{x}_j - E_j}{\bar{x}_j - \underline{x}_j}$, and

- $x_j = \bar{x}_j$ with probability $\bar{p}_j \stackrel{\text{def}}{=} \frac{E_j - \underline{x}_j}{\bar{x}_j - \underline{x}_j}$.

- *Algorithm:*

- simulate these distributions for x_j , $j < k$;
- simulate known distributions for $j > k$;
- use the simulated values $x_j^{(s)}$ to find

$$y^{(s)} = F(x_1^{(s)}, \dots, x_n^{(s)});$$

- sort N values $y^{(s)}$: $y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(N_i)}$;
- take $y_{(N_i \cdot (1-\varepsilon))}$ as y_0 .

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 35 of 53

Go Back

Full Screen

Close

Quit

35. Comment about Monte-Carlo Techniques

- *Traditional belief*: Monte-Carlo methods are inferior to analytical:
 - they are approximate;
 - they require large computation time;
 - simulations for *several* distributions, may mis-calculate the (desired) maximum over *all* distributions.
- *We proved*: the value corresponding to the selected distributions indeed provide the desired maximum value y_0 .
- *General comment*:
 - justified Monte-Carlo methods often lead to *faster* computations than analytical techniques;
 - example: multi-D integration – where Monte-Carlo methods were originally invented.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 36 of 53

Go Back

Full Screen

Close

Quit

36. Comment about Non-Linear Terms

- *Reminder:* in the above formula $D_i = a_i + \sum_{j=1}^n a_{ij} \cdot x_j$, we ignored quadratic and higher order terms in the dependence of each path time D_i on parameters x_j .
- *In reality:* we may need to take into account some quadratic terms.
- *Idea behind possible solution:* it is known that the $\max_i D = \max_i D_i$ of convex functions D_i is convex.
- *Condition when this idea works:* when each dependence $D_i(x_1, \dots, x_k, \dots)$ is still convex.
- *Solution:* in this case,
 - the function function D is still convex,
 - hence, our algorithm will work.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 37 of 53

Go Back

Full Screen

Close

Quit

37. Conclusions

- *Problem of chip design:* decrease the clock cycle.
- *How this problem is solved now:* by using worst-case (interval) techniques.
- *Limitations of this solution:* the probability of the worst-case values is usually very small.
- *Consequence:* estimates are over-conservative, hence over-design and under-performance of circuits.
- *Objective:* find the clock time as y_0 s.t. for the actual delay y , we have $\text{Prob}(y > y_0) \leq \varepsilon$ for given $\varepsilon > 0$.
- *Difficulty:* we only have *partial* information about the corresponding distributions.
- *What we have described:* a general technique that allows us, in particular, to compute y_0 .

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 38 of 53

Go Back

Full Screen

Close

Quit

38. Combining Interval and Probabilistic Uncertainty: General Case

- *Problem:* there are many ways to represent a probability distribution.
- *Idea:* look for an objective.
- *Objective:* make decisions $E_x[u(x, a)] \rightarrow \max_a$.
- *Case 1:* smooth $u(x)$.
- *Analysis:* we have $u(x) = u(x_0) + (x - x_0) \cdot u'(x_0) + \dots$
- *Conclusion:* we must know moments to estimate $E[u]$.
- *Case of uncertainty:* interval bounds on moments.
- *Case 2:* threshold-type $u(x)$.
- *Conclusion:* we need cdf $F(x) = \text{Prob}(\xi \leq x)$.
- *Case of uncertainty:* p-box $[\underline{F}(x), \overline{F}(x)]$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 39 of 53

Go Back

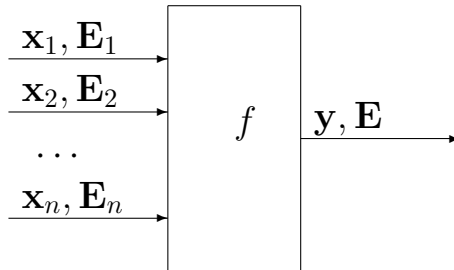
Full Screen

Close

Quit

39. Extension of Interval Arithmetic to Probabilistic Case: Successes

- *General solution:* parse to elementary operations $+$, $-$, $\cdot$, $1/x$, $\max$, $\min$.
- Explicit formulas for arithmetic operations known for intervals, for p-boxes $\mathbf{F}(x) = [\underline{F}(x), \overline{F}(x)]$, for intervals $+ 1\text{st moments } E_i \stackrel{\text{def}}{=} E[x_i]$:



General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 40 of 53

Go Back

Full Screen

Close

Quit

40. Successes (cont-d)

- *Easy cases*: $+$, $-$, product of independent x_i .
- *Example of a non-trivial case*: multiplication $y = x_1 \cdot x_2$, when we have no information about the correlation:
 - $\underline{E} = \max(p_1 + p_2 - 1, 0) \cdot \bar{x}_1 \cdot \bar{x}_2 + \min(p_1, 1 - p_2) \cdot \bar{x}_1 \cdot \underline{x}_2 + \min(1 - p_1, p_2) \cdot \underline{x}_1 \cdot \bar{x}_2 + \max(1 - p_1 - p_2, 0) \cdot \underline{x}_1 \cdot \underline{x}_2$;
 - $\bar{E} = \min(p_1, p_2) \cdot \bar{x}_1 \cdot \bar{x}_2 + \max(p_1 - p_2, 0) \cdot \bar{x}_1 \cdot \underline{x}_2 + \max(p_2 - p_1, 0) \cdot \underline{x}_1 \cdot \bar{x}_2 + \min(1 - p_1, 1 - p_2) \cdot \underline{x}_1 \cdot \underline{x}_2$,

where $p_i \stackrel{\text{def}}{=} (E_i - \underline{x}_i) / (\bar{x}_i - \underline{x}_i)$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 41 of 53

Go Back

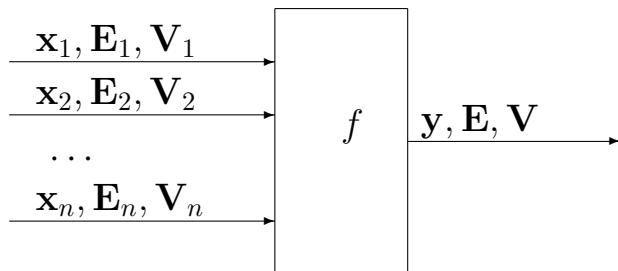
Full Screen

Close

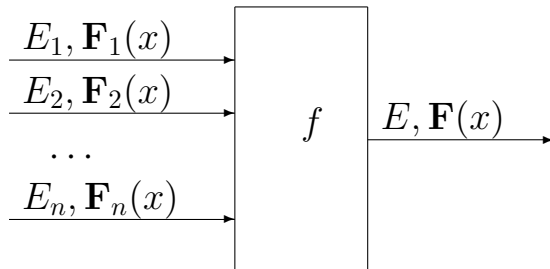
Quit

41. Challenges

- intervals + 2nd moments:



- moments + p-boxes; e.g.:



General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 42 of 53

Go Back

Full Screen

Close

Quit

42. Case Study: Bioinformatics

- *Practical problem:* find genetic difference between cancer cells and healthy cells.
- *Ideal case:* we directly measure concentration c of the gene in cancer cells and h in healthy cells.
- *In reality:* difficult to separate.
- *Solution:* we measure $y_i \approx x_i \cdot c + (1 - x_i) \cdot h$, where x_i is the percentage of cancer cells in i -th sample.
- *Equivalent form:* $a \cdot x_i + h \approx y_i$, where $a \stackrel{\text{def}}{=} c - h$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 43 of 53

Go Back

Full Screen

Close

Quit

43. Case Study: Bioinformatics (cont-d)

- If we know x_i exactly: Least Squares Method

$$\sum_{i=1}^n (a \cdot x_i + h - y_i)^2 \rightarrow \min_{a,h}, \text{ hence } a = \frac{C(x,y)}{V(x)} \text{ and}$$

$$h = E(y) - a \cdot E(x), \text{ where } E(x) = \frac{1}{n} \cdot \sum_{i=1}^n x_i,$$

$$V(x) = \frac{1}{n-1} \cdot \sum_{i=1}^n (x_i - E(x))^2,$$

$$C(x,y) = \frac{1}{n-1} \cdot \sum_{i=1}^n (x_i - E(x)) \cdot (y_i - E(y)).$$

- *Interval uncertainty*: experts manually count x_i , and only provide interval bounds $\mathbf{x}_i$, e.g., $x_i \in [0.7, 0.8]$.
- *Problem*: find the range of a and h corresponding to all possible values $x_i \in [\underline{x}_i, \bar{x}_i]$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀◀

▶▶

◀

▶

Page 44 of 53

Go Back

Full Screen

Close

Quit

44. General Problem

- *General problem:*
 - we know intervals $\mathbf{x}_1 = [\underline{x}_1, \overline{x}_1], \dots, \mathbf{x}_n = [\underline{x}_n, \overline{x}_n]$,
 - compute the range of $E(x) = \frac{1}{n} \sum_{i=1}^n x_i$, population variance $V = \frac{1}{n} \sum_{i=1}^n (x_i - E(x))^2$, etc.
- *Difficulty:* NP-hard even for variance.
- *Known:*
 - efficient algorithms for $\underline{V}$,
 - efficient algorithms for $\overline{V}$ and $C(x, y)$ for reasonable situations.
- *Bioinformatics case:* find intervals for $C(x, y)$ and for $V(x)$ and divide.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 45 of 53

Go Back

Full Screen

Close

Quit

45. Case Study: Detecting Outliers

- In many application areas, it is important to detect *outliers*, i.e., unusual, abnormal values.
- In *medicine*, unusual values may indicate disease.
- In *geophysics*, abnormal values may indicate a mineral deposit (or an erroneous measurement result).
- In *structural integrity* testing, abnormal values may indicate faults in a structure.
- *Traditional engineering approach*: a new measurement result x is classified as an outlier if $x \notin [L, U]$, where

$$L \stackrel{\text{def}}{=} E - k_0 \cdot \sigma, \quad U \stackrel{\text{def}}{=} E + k_0 \cdot \sigma,$$

and $k_0 > 1$ is pre-selected.

- *Comment*: most frequently, $k_0 = 2, 3$, or 6 .

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page



Page 46 of 53

Go Back

Full Screen

Close

Quit

46. Outlier Detection Under Interval Uncertainty: A Problem

- In some practical situations, we only have intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$.
- Different $x_i \in \mathbf{x}_i$ lead to different intervals $[L, U]$.
- A *possible* outlier: outside *some* k_0 -sigma interval.
- *Example*: structural integrity – not to miss a fault.
- A *guaranteed* outlier: outside *all* k_0 -sigma intervals.
- *Example*: before a surgery, we want to make sure that there is a micro-calcification.
- A value x is a possible outlier if $x \notin [\overline{L}, \underline{U}]$.
- A value x is a guaranteed outlier if $x \notin [\underline{L}, \overline{U}]$.
- *Conclusion*: to detect outliers, we must know the ranges of $L = E - k_0 \cdot \sigma$ and $U = E + k_0 \cdot \sigma$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀ ▶

◀ ▶

Page 47 of 53

Go Back

Full Screen

Close

Quit

47. Outlier Detection Under Interval Uncertainty: A Solution

- *We need:* to detect outliers, we must compute the ranges of $L = E - k_0 \cdot \sigma$ and $U = E + k_0 \cdot \sigma$.
- *We know:* how to compute the ranges $\mathbf{E}$ and $[\underline{\sigma}, \overline{\sigma}]$ for E and σ .
- *Possibility:* use interval computations to conclude that $L \in \mathbf{E} - k_0 \cdot [\underline{\sigma}, \overline{\sigma}]$ and $U \in \mathbf{E} + k_0 \cdot [\underline{\sigma}, \overline{\sigma}]$.
- *Problem:* the resulting intervals for L and U are *wider* than the actual ranges.
- *Reason:* E and σ use the same inputs $x_1, \dots, x_n$ and are hence not independent from each other.
- *Practical consequence:* we miss some outliers.
- *Desirable:* compute *exact* ranges for L and U .
- *Application:* detecting outliers in gravity measurements.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 48 of 53

Go Back

Full Screen

Close

Quit

48. Acknowledgments

This work was supported in part by:

- by National Science Foundation grants HRD-0734825, EAR-0225670, and EIA-0080940, and
- by Texas Department of Transportation grant No. 0-5453.

General Problem of...

Interval Arithmetic:...

Case Study: Chip Design

Combining Interval...

Case Study:...

Case Study: Detecting...

Acknowledgments

Fuzzy Computations:...

Title Page



Page 49 of 53

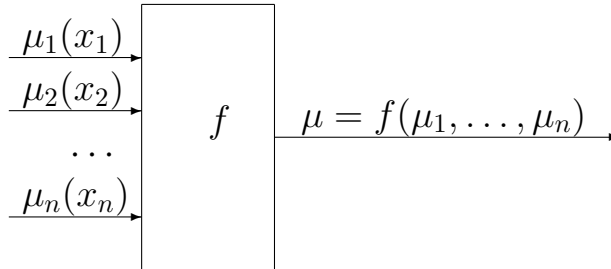
Go Back

Full Screen

Close

Quit

49. Fuzzy Computations: A Problem



- *Given:* an algorithm $y = f(x_1, \dots, x_n)$ and n fuzzy numbers $\mu_i(x_i)$.
- *Compute:* $\mu(y) = \max_{x_1, \dots, x_n: f(x_1, \dots, x_n) = y} \min(\mu_1(x_1), \dots, \mu_n(x_n))$.
- *Motivation:* y is a possible value of $Y \leftrightarrow \exists x_1, \dots, x_n$ s.t. each x_i is a possible value of X_i and $f(x_1, \dots, x_n) = y$.
- *Details:* “and” is $\min$, $\exists$ (“or”) is $\max$, hence
$$\mu(y) = \max_{x_1, \dots, x_n} \min(\mu_1(x_1), \dots, \mu_n(x_n), t(f(x_1, \dots, x_n) = y)),$$
where $t(\text{true}) = 1$ and $t(\text{false}) = 0$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 50 of 53

Go Back

Full Screen

Close

Quit

50. Fuzzy Computations: Reduction to Interval Computations

- *Problem (reminder):*
 - *Given:* an algorithm $y = f(x_1, \dots, x_n)$ and n fuzzy numbers X_i described by membership functions $\mu_i(x_i)$.
 - *Compute:* $Y = f(X_1, \dots, X_n)$, where Y is defined by Zadeh's extension principle:

$$\mu(y) = \max_{x_1, \dots, x_n: f(x_1, \dots, x_n) = y} \min(\mu_1(x_1), \dots, \mu_n(x_n)).$$

- *Idea:* represent each X_i by its α -cuts

$$X_i(\alpha) = \{x_i : \mu_i(x_i) \geq \alpha\}.$$

- *Advantage:* for continuous f , for every α , we have

$$Y(\alpha) = f(X_1(\alpha), \dots, X_n(\alpha)).$$

- *Resulting algorithm:* for $\alpha = 0, 0.1, 0.2, \dots, 1$ apply interval computations techniques to compute $Y(\alpha)$.

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 51 of 53

Go Back

Full Screen

Close

Quit

51. Proof of the Result about Chips

- Let us fix the optimal distributions for $x_2, \dots, x_n$; then,

$$\text{Prob}(D \leq y_0) = \sum_{(x_1, \dots, x_n): D(x_1, \dots, x_n) \leq y_0} p_1(x_1) \cdot p_2(x_2) \cdot \dots$$

- So, $\text{Prob}(D \leq y_0) = \sum_{i=0}^N c_i \cdot q_i$, where $q_i \stackrel{\text{def}}{=} p_1(v_i)$.
- Restrictions: $q_i \geq 0$, $\sum_{i=0}^N q_i = 1$, and $\sum_{i=0}^N q_i \cdot v_i = E_1$.
- Thus, the worst-case distribution for x_1 is a solution to the following linear programming (LP) problem:

$$\begin{aligned} &\text{Minimize } \sum_{i=0}^N c_i \cdot q_i \text{ under the constraints } \sum_{i=0}^N q_i = 1 \text{ and} \\ &\sum_{i=0}^N q_i \cdot v_i = E_1, \quad q_i \geq 0, \quad i = 0, 1, 2, \dots, N. \end{aligned}$$

General Problem of ...

Interval Arithmetic: ...

Case Study: Chip Design

Combining Interval ...

Case Study: ...

Case Study: Detecting ...

Acknowledgments

Fuzzy Computations: ...

Title Page

◀

▶

◀

▶

Page 52 of 53

Go Back

Full Screen

Close

Quit

52. Proof of the Result about Chips (cont-d)

- *Minimize:* $\sum_{i=0}^N c_i \cdot q_i$ under the constraints $\sum_{i=0}^N q_i = 1$ and $\sum_{i=0}^N q_i \cdot v_i = E_1$, $q_i \geq 0$, $i = 0, 1, 2, \dots, N$.
- *Known:* in LP with $N + 1$ unknowns $q_0, q_1, \dots, q_N$, $\geq N + 1$ constraints are equalities.
- *In our case:* we have 2 equalities, so at least $N - 1$ constraints $q_i \geq 0$ are equalities.
- Hence, no more than 2 values $q_i = p_1(v_i)$ are non-0.
- If corresponding v or v' are in $(\underline{x}_1, \bar{x}_1)$, then for $[v, v'] \subset \mathbf{x}_1$ we get the same y_0 – in contradiction to non-degeneracy.
- Thus, the worst-case distribution is located at $\underline{x}_1$ and $\bar{x}_1$.
- The condition that the mean of x_1 is E_1 leads to the desired formulas for $\underline{p}_1$ and $\bar{p}_1$.

[General Problem of ...](#)[Interval Arithmetic: ...](#)[Case Study: Chip Design](#)[Combining Interval ...](#)[Case Study: ...](#)[Case Study: Detecting ...](#)[Acknowledgments](#)[Fuzzy Computations: ...](#)[Title Page](#)[◀◀](#)[▶▶](#)[◀](#)[▶](#)[Page 53 of 53](#)[Go Back](#)[Full Screen](#)[Close](#)[Quit](#)