Why Daubechies wavelets are so successful

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1. Need for 1-D wavelets

- The values of most physical quantities q change with time t: $q = q(t)$.
- In some cases, e.g., in celestial mechanics, we know the general shape of this dependence.
- In other words, we know that the function $f(t, c_1, \ldots, c_n)$ such that:
  - the actual dynamics $q(t)$
  - is determined by this expression $q(t) = f(t, c_1, \ldots, c_n)$ for some values of the parameters $c_1, \ldots, c_n$.
- In such cases, first, we use the values $q(t_1)$, ..., $q(t_k)$ observed during some observation period to determine the values of these parameters.
- I.e., we solve the system of equations

$$q(t_i) = f(t_i, c_1, \ldots, c_n), \quad i = 1, \ldots, k, \text{ with the unknowns } c_i.$$

- Then, we use the resulting values c_i to predict the future values $q(t)$ of the quantity q as $q(t) = f(t, c_1, \ldots, c_n)$.

2. Need for 1-D wavelets (cont-d)

- Sometimes, the dependence on the parameters c_i is non-linear.
- Then this system of equations is not easy to solve.
- However, if this is how q depends on time, there is nothing we can do about it.
- In many other cases, however, we do not know the shape of the dependence.
- In such cases, it is also desirable:
  - to come up with a general formula $f(t, c_1, \ldots, c_n)$ – with not-too-many parameters c_i,
  - that would adequately describe the dynamics of the quantity of interest.

3. Need for 1-D wavelets (cont-d)

- In such situations, it is reasonable to select a family for which the corresponding system of equations:
  - is the easiest to solve,
  - i.e., is a system of linear equations.
- For this purpose, we select a family for which the dependence $f(t, c_1, \ldots, c_n)$ is linear in terms of the unknowns:

  $f(t, c_1, \ldots, c_n) = f_0(t) + c_1 \cdot f_1(t) + \ldots + c_n \cdot f_n(t)$ for some functions $f_i(t)$.
- Such representations are indeed actively used in data processing.
- For example, for a smooth dependence $q(t)$:
  - it is reasonable to approximate it by a polynomial,
  - i.e., by the sum of the first few terms of its Taylor expansion.
- In this case, the functions $f_i(t)$ are monomials t^j corresponding to non-negative integers j.

4. Need for 1-D wavelets (cont-d)

- For a periodic process with a known period T, we can use sines and cosines $\sin(j \cdot \omega \cdot t)$ and $\cos(j \cdot \omega \cdot t)$ for non-negative integers j.
- Here, $\omega = 2\pi/T$.
- This is known as *Fourier series*.
- However, many physical processes – e.g., seismic processes – are neither smooth nor periodical.
- They consist of time-localized bursts of activity.
- To describe such processes, it makes sense to use similarly time-localized functions $f_i(t)$.
- Such functions are known as *wavelets*.

5. From generic wavelets to Daubechies wavelets

- One of the computational advantages of Fourier series is that:
  - all the functions $f_i(t)$ used in this approximation
  - can be obtained from each other by scaling and shift,
  - i.e., they all have the form $f_i(t) = f_0(a_i \cdot t + b_i)$ for some values a_i and b_i, where $f_0(t) \stackrel{\text{def}}{=} \sin(\omega \cdot t)$.
- It turns out that we can select wavelets that satisfy a similar property; namely:
  - we select the basic function $\varphi(t)$ known as the *mother wavelet*, and
  - the functions $f_i(t)$ of the form $\varphi(2^j \cdot t - \ell)$, where $j \geq 0$ and ℓ are integers.
- There are also similar functions generated by another related function, known as the *father wavelet*.

6. From generic wavelets to Daubechies wavelets (cont-d)

- We want the resulting functions $f_i(t)$ to be efficient in representing and processing data.
- So, the mother wavelet must satisfy a certain linear functional equation.
- This functional equation has many different solutions; empirically:
  - wavelets corresponding to some solutions work better, while
  - wavelet corresponding to other solutions of the functional equations do not work so well.
- To select a single solution – and thus, to fix a family of wavelets – we need to impose additional restrictions on the function $\varphi(t)$.
- We want to make computations easier.
- We also want to preserve linearity of the corresponding system of equation.

7. From generic wavelets to Daubechies wavelets (cont-d)

- So, it makes sense to impose restrictions which are linear in terms of $\varphi(t)$.
- A general such linear restriction has the form

$$\int c_m(t) \cdot \varphi(t)\, dt = b_m, \quad m = 1, \ldots, M, \text{ for some function } c_m(t).$$

- Once we know one solution $\varphi_0(t)$ to the system of linear equations, we can have an even simpler system of linear equations for the difference

$$\Delta(t) \stackrel{\text{def}}{=} \varphi(t) - \varphi_0(t) : \quad \int c_m(t) \cdot \Delta(t)\, dt = 0, \quad m = 1, \ldots, M.$$

8. From generic wavelets to Daubechies wavelets (cont-d)

- Of course:
  - once the equality $\int c(t)\cdot\Delta(t)\,dt = 0$ holds for all M functions $c_1(t), \ldots, c_M(t)$,
  - the same equality holds for all possible linear combinations $c(t) = s_1\cdot c_1(t) + \ldots + s_M\cdot c_M(t)$ of these functions,
  - i.e., for the whole M-dimensional linear space L of functions generated by the functions $c_m(t)$.
- From this viewpoint, the above condition can be described as requiring that $\int c(t)\cdot\Delta(t)\,dt = 0$ for all functions $c(t)$ from L.
- Thus, the selection of a specific wavelet family means selecting an appropriate linear space L.

9. From generic wavelets to Daubechies wavelets (cont-d)

- Ingrid Daubechies proposed:
  - to use $c_m(t) = t^{m-1}$,
  - i.e., equivalently, to take, as L, the linear space of all polynomials $c(t) = a_0 + a_1 \cdot t + \ldots + a_{M-1} \cdot t^{M-1}$ of order less than M.
- The resulting wavelets are known as *Daubechies wavelets.*

10. Empirical fact

- In many practical applications – in particular, in processing seismic signals:
  - Daubechies wavelets work very well,
  - much better than many other wavelet families.
- In this talk, we provide a possible theoretical explanation for this empirical success.
- Namely, we show that these wavelets are optimal with respect to any scale- and shift-invariant optimality criterion.

11. We need to select an M-dimensional linear space

- As we have mentioned, selecting a family of wavelets is equivalent to selecting an M-dimensional linear space L of functions.
- In these terms, the question is:

What is the optimal selection of an M-dimensional linear space of functions?

12. We will only consider smooth (differentiable) functions $c(t)$

- In wavelet analysis, the corresponding functions $c(t)$ are differentiable.
- In view of this, in this talk, we will also limit ourselves to the case when all the functions $a(t)$ from the linear space L are differentiable.
- This diffentiability requirement makes sense.
- E.g., it is known that every continuous function $c(t)$ can be approximated, with any given accuracy, by smooth functions.
- From the practical viewpoint, a very small difference is not noticeable.
- It thus makes sense to assume that all the functions $c(t)$ are differentiable.

13. Smooth functions: caution

- It is worth noticing that it is not possible to follow this argument too far.
- Actually, some wavelets are *not* smooth.
- Even Daubechies wavelets of higher order M, while smooth, are not infinitely differentiable:
  - if we differentiate the corresponding mother wavelet again and again,
  - we will eventually reach a function which is not differentiable at some points.

14. What does "optimal" mean

- Usually, when we say that an alternative $A_{\rm opt}$ is optimal, it means that:
  - there is a numerical characteristic $F(A)$ describing the imperfection of different alternatives, and
  - the alternative $A_{\rm opt}$ has the smallest value of this characteristic.
- For example, for different wavelet families A, we can take, as $F(A)$ the mean squared accuracy with which:
  - the use of the first few wavelets from this family
  - approximates signals from the given set of signals.
- However, this is not the only way to describe optimality.
- In the above example, we may have several different families with the same smallest possible value of the mean squared accuracy.
- In such a case, we can use this non-uniqueness to minimize some other characteristic $G(A)$.

15. What does "optimal" mean (cont-d)

- E.g., we can minimize the average computation time needs to get the corresponding approximation.
- We then say that the alternative A is better or of the same quality as an alternative B – we will denote it by $A \le B$ – if:
  - either $F(A) < F(B)$,
  - or $F(A) = F(B)$ and $G(A) \le G(B)$.
- If this additional numerical criterion does not lead to a unique selection of an alternative, we can minimize something else, etc.
- At the end, we reach the *final* optimality criterion – for which there is exactly one optimal alternative.

16. What does "optimal" mean (cont-d)

- No matter how complex our comparison, in all these cases, we have a relation $A \leq B$ between the two alternatives.
- This relation describes that A is better or of the same quality as B.
- Of course, each alternative has the same quality as itself $A \leq A$, and if $A \leq B$ and $B \leq C$, then $A \leq C$.
- Thus, we arrive at the following definition.

17. Resulting definition

- Let $\mathcal{A}$ be a set. Its elements will be called *alternatives*.
- By an *optimality criterion*, we mean a binary relation $\le$ on this set which satisfies the following two properties:
  - for every $A \in \mathcal{A}$, we have $A \le A$ (*reflexivity*), and
  - for all $A, B, C \in \mathcal{A}$, if $A \le B$ and $B \le C$, then $A \le C$.
- An alternative $A_{\rm opt}$ is called *optimal* with respect to the optimal criterion $\le$ if we have $A_{\rm opt} \le A$ for all $A \in \mathcal{A}$.
- An optimality criterion $\le$ is called *final* if for this criterion, there exists exactly one optimal alternative.

18. Natural invariance properties

- We are interested in describing how a quantity changes with time.
- We describe this dependence in numerical terms, as a dependence $q(t)$:
  - of the numerical value of the quantity q
  - on the numerical value of time t.
- However, the numerical value of time depends on the selection of the measuring unit and on the selection of the starting point.
- We can replace the original unit with a new one which is a times smaller – e.g., consider seconds instead of minutes.
- Then all numerical values of time are multiplied by a.
- The corresponding linear transformation $t \mapsto a \cdot t$ is known as *scaling*.
- Similarly, we can replace the original starting point for measuring time with a new starting point which is t_0 moments earlier.

19. Natural invariance properties (cont-d)

- Then this value t_0 will be added to all numerical values of time.
- The corresponding linear transformation $t \mapsto t + t_0$ is known as *shift*.
- In general, if we change both the unit and the starting point, we replace t with $a \cdot t + t_0$.
- So, we get a linear transformation.
- The numerical values change, but the physical process remains the same.
- From this viewpoint, it is reasonable to require that:
  - the relative quality of two different methods should not change
  - if we simply change the unit and/or the starting point.
- In terms of linear spaces – that describe different wavelet families – we thus arrive at the following definition.

20. Resulting definition

- By a *linear transformation*, we mean a function $T(t) = a \cdot t + t_0$ for some values a and t_0.
- For each linear transformation T and each function $e(t)$, by the *result* $T(e)$ of applying T to e we mean a function $e(T(t))$.
- Let L be an M-dimensional linear space of smooth functions.
- By the *result* $T(L)$ of applying T to L we mean the linear space formed by the functions $T(e)$ for $e \in L$.
- We say that the optimality criterion $\leq$ on the set $\mathcal{L}$ of all M-dimensional linear spaces of smooth functions is *invariant* if:

  for every two spaces, $L_1 \leq L_2$ implies that $T(L_1) \leq T(L_2)$.

- Now, we can formulate our main result.

21. Main result

- Let $\leq$ be a final invariant optimality criterion on the set of all M-dimensional linear spaces of smooth functions.
- Then all elements of the optimal family $L_{\rm opt}$ are polynomials of order less than M.
- Thus, we have indeed proven that the linear space corresponding to Daubechies wavelets is optimal.
- So, in this sense, Daubechies wavelets are optimal.

22. Proof

- Let us first prove that the optimal family $L_{\rm opt}$ is itself invariant, i.e., that $T(L_{\rm opt})=L_{\rm opt}$.
- Indeed, the fact that $L_{\rm opt}$ is optimal means that $L_{\rm opt}\le L$ for all families L.
- In particular, this is true for all families of the type $T^{-1}(L)$, where T^{-1} is the inverse transformation.
- So, $L_{\rm opt}\le T^{-1}(L)$ for each L.
- By using invariance of the optimality criterion, we conclude that $T(L_{\rm opt})\le L$ for every L.
- So, the linear space $T(L_{\rm opt})$ is also optimal.
- However, the optimality criterion $\le$ is final, which means that there is only one optimal space, so indeed, $T(L_{\rm opt})=L_{\rm opt}$.
- Let us now select any basis $e_1(t)$, $\dots$, $e_M(t)$ in the optimal linear space.

23. Proof (cont-d)

- Invariance of the linear space L_{opt} means, in particular, that:
  - for each i and for each t_0,
  - the shifted function $e_i(t+t_0)$ also belongs to this linear space.
- So, $e_i(t+t_0)=\sum\limits_{j=1}^{M} C_{ij}(t_0)\cdot e_j(t)$ for some C_{ij} depending on t_0.
- Let us select M different moments of time $t_1,\ldots,t_M$, then

$$e_i(t_k+t_0)=\sum_{j=1}^{M} C_{ij}(t_0)\cdot e_j(t_k),\quad k=1,\ldots,M.$$

- We get a system of M linear equations to determine the coefficients $C_{ij}(t_0)$ in terms of the functions e_j.
- In general, the solution of a system of linear equations is a linear combination of the left-hand sides.
- The left-hand sides $e_i(t_k+t_0)$ are differentiable functions of t_0.

24. Proof (cont-d)

- Thus, all the coefficients $C_{ij}(t_0)$ are also differentiable.
- So, all the functions in the above formula are differentiable.
- Thus, we can differentiate both sides with respect to t_0, and get

$$e_i'(t+t_0)=\sum_{j=1}^{M} C_{ij}'(t_0)\cdot e_j(t).$$

- In particular, for $t_0=0$, we get $e_i'(t)=\sum\limits_{j=1}^{M} c_{ij}\cdot e_j(t)$, where we denoted $c_{ij}\stackrel{\text{def}}{=}C_{ij}'(0)$.
- So, for M functions $e_1(t)$, $\ldots$, $e_M(t)$, we have a system of M linear differential equations with constant coefficients.

25. Proof (cont-d)

- It is known that a general solution to such a system is a linear combination of expressions of the type $t^k \cdot \exp(\alpha \cdot t)$, where:
  - the value α is an eigenvalue of the matrix $\|c_{ij}\|$, and
  - the value k is an non-negative integer which is smaller that the multiplicity of this eigenvalue.
- Similarly, invariance implies that for every i and for every a, the function $e_i(a \cdot t)$ also belongs to the optimal space $L_{\rm opt}$.
- So, $e_i(a \cdot t) = \sum\limits_{j=1}^{M} C_{ij}(a) \cdot e_j(t)$ for some C_{ij} depending on a.
- If we select M different moments of time $t_1, \ldots, t_M$, we get a system of M linear equations to determine these coefficients in terms of e_j:

$$e_i(a \cdot t_k) = \sum_{j=1}^{M} C_{ij}(a) \cdot e_j(t_k), \quad k = 1, \ldots, M.$$

26. Proof (cont-d)

- In general, the solution of a system of linear equations is a linear combination of the left-hand sides.
- The left-hand sides $e_i(a \cdot t_k)$ are differentiable functions of t_0.
- Thus, all the dependence of all the coefficients $C_{ij}(a)$ is also differentiable.
- So, all the functions in the above formula are differentiable.
- Thus, we can differentiate both sides with respect to a, and get

$$t \cdot e_i'(a \cdot t) = \sum_{j=1}^{M} C_{ij}'(a) \cdot e_j(t).$$

- In particular, for $a = 1$, we get

$$t \cdot e_i'(t) = \sum_{j=1}^{M} c_{ij} \cdot e_j(t), \text{ where we denoted } c_{ij} \stackrel{\text{def}}{=} C_{ij}'(1).$$

27. Proof (cont-d)

- Let us introduce an auxiliary variable $x \stackrel{\text{def}}{=} \ln(t)$, so that $t = \exp(x)$ and $dx = dt/t$; then:

$$t \cdot \frac{de_i}{dt} = \frac{de_i}{dt/t} = \frac{de_i}{dx}.$$

- So, the above formula takes the form

$$\frac{dE_i(x)}{dx} = \sum_{j=1}^{M} c_{ij} \cdot E_j(x), \text{ where } E_i(x) \stackrel{\text{def}}{=} e_i(\exp(x)).$$

- So, for M functions $E_1(x), \ldots, E_M(x)$, we also have a system of M linear differential equations with constant coefficients.

- Thus, each of these functions is a linear combination of the expressions of the type $x^k \cdot \exp(\alpha \cdot x)$.

- So, each function $e_i(t) = E_i(\ln(x))$ is a linear combination of functions

$$(\ln t)^k \cdot \exp(\alpha \cdot \ln(t)) = (\ln t)^k \cdot t^\alpha.$$

28. Proof (cont-d)

- We want a function representable both as a linear combination of these expressions and a linear combination of expressions $t^k \cdot \exp(\alpha \cdot t)$.
- So, we have $k = 0$ and α must be an integer.
- So, each function $e_i(t)$ must be a linear combination of monomials t^k – i.e., a polynomial.
- To complete the proof, let us show that all polynomials can only have degree $< M$.
- Indeed, suppose that the optimal linear space $L_{\rm opt}$ contains a polynomial of degree d, i.e., a function

$$e^{(0)}(t) = a_0 \cdot t^d + a_1 \cdot t^{d-1} + \ldots, \text{ with } a_0 \neq 0.$$

- The optimal linear space is invariant with respect to shift.
- So for each h, the function $e^{(0)}(t + h)$ also belong to this space.

29. Proof (cont-d)

- Since $L_{\rm opt}$ is a linear space, it also contains any linear combination of the two functions $e^{(0)}(t)$ and $e^{(0)}(t+h)$, in particular, their difference

$$e^{(1)}(t) \stackrel{\rm def}{=} e^{(0)}(t+1) - e^{(0)}(t).$$

- One can check that this difference is a polynomial

$$e^{(1)}(t) = d \cdot a_0 \cdot t^{d-1} + \ldots \text{ of degree } \le (d-1).$$

- By applying this difference again and again, we get a polynomial $e^{(2)}(t) = e^{(1)}(t+1) - e^{(1)}(t)$ of degree $\le (d-2)$, etc.
- This leads all the way to a polynomial $e^{(d)}(t) = e^{(d-1)}(t+1) - e^{(d-1)}(t)$ of degree 0, i.e., to a constant.
- These $d+1$ polynomials $e^{(0)}(t), \ldots, e^{(d)}(t)$ are all linearly independent.

30. Proof (cont-d)

- Indeed, each linear combination $c_{i_1} \cdot e^{(i_1)}(t) + \ldots + c_{i_k} \cdot e^{(i_k)}(t)$ for some $i_1 < i_2 < \ldots$ and all $c_{i_j} \neq 0$ starts with a non-zero term $\sim t^{d-i_1}$.
- Thus, it cannot be identically 0.
- According to linear algebra, in an M-dimensional space, we can have no more than M linearly independent elements.
- So here we have $d + 1 \leq M$, thus $d \leq M - 1$, hence indeed $d < M$.
- The proposition is proven.

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