How to Decide Which Cracks Should be Repaired First: Theoretical Explanation of Empirical Formulas

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1. Which Cracks Should Be Repaired First?

- Under stress, cracks appear in constructions.
- They appear in buildings, they appear in brides, they appear in pavements, they appear in engines, etc.
- Once a crack appears, it starts growing.
- Cracks are potentially dangerous.
- Cracks in an engine can lead to a catastrophe.
- Cracks in a pavement makes a road more dangerous and prone to accidents, etc.
- It is therefore desirable to repair the cracks.
- In the ideal world, each crack should be repaired as soon as it is noticed.
- This is indeed done in critical situations.

2. Which Cracks to Repair First (cont-d)

- Example: after each flight, the Space Shuttle was thoroughly studied and all cracks were repaired.
- In less critical situations, for example, in pavement engineering, our resources are limited.
- In such situations, we need to decide which cracks to repair first.
- We must concentrate efforts on cracks that, if unrepaired, will become most dangerous.
- For that, we need to be able to predict how each crack will grow, e.g., in the next year:
  - once we are able to predict how the current cracks will grow,
  - we will be able to concentrate our limited repair resources on most potentially harmful cracks.

3. To Make a Proper Decision, It Is Desirable to Have Theoretically Justified Formulas

- Crack growth is a very complex problem, it is very difficult to analyze theoretically.
- So far, first-principle-based computer models have not been very successful in describing crack growth.
- Good news is that cracks are ubiquitous.
- There is a lot of empirical data about the crack growth.
- Researchers have come up with empirical approximate formulas.
- In the following, we will describe the state-of-the-art empirical formulas.
- However, purely empirical formulas are not always reliable.

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4. Need for Theoretical Formulas (cont-d)

- There have been many cases when an empirical formula turned out to be:
  - true only in limited cases,
  - and false in many others.
- Even the great Newton naively believed that:
  - since the price of a certain stock was growing exponentially for some time, it will continue growing,
  - so he invested all his money in that stock and lost almost everything when the bubble collapsed.

5. Need for Theoretical Formulas (cont-d)

- From this viewpoint:
  - taking into account that missing a potentially dangerous crack can be catastrophic,
  - it is desirable to have theoretically justified formulas for crack growth.
- This is what we do in this talk: we provide theoretical explanations for the existing empirical formulas.
- With this goal in mind, let us recall the main empirical formulas for crack growth.

6. How Cracks Grow: A General Description

- In most cases, stress comes in cycles:
  - the engine clearly goes through the cycles,
  - the road segment gets stressed when a vehicle passes through it, etc.
- So, the crack growth is usually expressed by describing:
  - how the length a of the crack changes
  - during a stress cycle at which the stress is equal to some value σ.
- The increase in length is usually denoted by Δa.
- So, to describe how a crack grows, we need to find out how Δa depends on a and σ: $\Delta a = f(a, \sigma)$.

7. Case of Very Short Cracks

- The first empirical formula – known as Wöhler law – was proposed to describe how cracks appear.
- In the beginning, the length a is 0 (or very small).
- So the dependence on a can be ignored, and we have $\Delta a = f(\sigma)$, for some function $f(\sigma)$.
- Empirical data shows that this dependence is a power law: $\Delta a = C_0 \cdot \sigma^{m_0}$, for some constants C_0 and m_0.

8. Practical Case of Reasonable Size Cracks

- In critical situations, the goal is to prevent the cracks from growing.
- In such situations, very small cracks are extremely important.
- In most other practical viewpoint, small cracks are usually allowed to grow.
- So the question is how cracks of reasonable size grow.
- Several empirical formulas have been proposed.
- In 1963, P. C. Paris and F. Erdogan compared all these formulas with empirical data.
- They came up with a new empirical formula that best fits the data: $\Delta a = C \cdot \sigma^m \cdot a^{m'}$.
- This *Paris Law* (aka *Paris-Erdogan Law*) is still in use.

9. Usual Case of Paris Law

- Usually, $m' = m/2$, so $\Delta a = C\cdot\sigma^m\cdot a^{m/2} = C\cdot(\sigma\cdot\sqrt{a}\,)^m$.
- Paris formula is empirical, but the dependence $m' = m/2$ has theoretical explanations.
- One of such explanations is that the stress acts randomly at different parts of the crack.
- According to statistics, on average, the effect of n independent factors is proportional to $\sqrt{n}$.
- A crack of length a consists of a/δ_a independent parts.
- So, the overall effect K of the stress σ is proportional to $K = \sigma\cdot\sqrt{n} \sim \sigma\cdot\sqrt{a}$.
- This quantity K is known as *stress intensity*.
- For the power law $\Delta a = C\cdot K^m$, this leads to $\Delta a = \text{const}\cdot(\sigma\cdot\sqrt{a}\,)^m = \text{const}\cdot\sigma^m\cdot a^{m/2}$, i.e., to $m' = m/2$.

10. Empirical Dependence Between C and m

- In principle, we can have all possible combinations of C and m.
- Empirically, however, there is a relation between C and m: $C = c_0 \cdot b_0^m$.

11. Beyond Paris Law

- As we have mentioned, Paris law is only valid for reasonably large crack lengths a.
- It cannot be valid for $a = 0$.
- Indeed, for $a = 0$, it implies that $\Delta a = 0$ and thus, that cracks cannot appear by themselves.
- However, cracks do appear.
- It was proposed to use Paris Law with different values of C, m, and m' for different ranges of a.
- This worked OK, but not perfectly.
- The best empirical fit came from the following generalization of Paris law:

$$\Delta a = C \cdot \sigma^m \cdot \left(a^\alpha + c \cdot \sigma^\beta\right)^\gamma .$$

- Empirically, we have $\alpha \approx 1$.

12. What We Do in This Talk

- In this talk, we provide a theoretical explanation for the above empirical formulas.
- Our explanations use the general ideas of scale-invariance.
- Similar ideas have been used in the past to explain Paris law.
- The existence of theoretical explanations makes us confident that:
  - the current empirical formulas
  - can (and should) be used in the design of the corresponding automatic decision systems.

13. Scale Invariance: Main Idea

- In general, we want to find the dependence $y = f(x)$ of one physical quantity on another one.
- E.g., for short cracks, the dependence of crack growth on stress.
- When we analyze the data, we deal with numerical values of these quantities.
- However, the numerical values depend on the selection of the measuring unit; for example:
  - if we measure crack length in centimeters,
  - we get numerical values which are 2.54 times larger than if we use inches.

14. Scale Invariance (cont-d)

- In general:
  - if we replace the original measuring unit with a new unit which is λ times smaller,
  - all the numerical values get multiplied by λ: instead of the original value x, we get a new value $x' = \lambda \cdot x$.
- In many physical situations, there is no preferred measuring unit.
- In such situations, it makes sense to require that the dependence $y = f(x)$ remain valid in all possible units.
- Of course, if we change a unit for x, then we need to appropriately change the unit for y.

15. Scale Invariance (cont-d)

- So the corresponding *scale invariance* requirement takes the following form.
- For every $\lambda > 0$, there exists a value $\mu(\lambda)$ depending on λ such that:
  - if we have $y = f(x)$,
  - then in the new units $y' = \mu(\lambda) \cdot y$ and $x' = \lambda \cdot x$, we should have $y' = f(x')$.
- For the dependence $y = f(x_1, \ldots, x_v)$ on several quantities $x_1, \ldots, x_v$:
  - for all possible tuples $(\lambda_1, \ldots, \lambda_v)$,
  - there should exist a value $\mu(\lambda_1, \ldots, \lambda_v)$ such that
  - if we have $y = f(x_1, \ldots, x_v)$, then
  - in the new units $x_i' = \lambda_i \cdot x_i$ and $y' = \mu(\lambda_1, \ldots, \lambda_v) \cdot y$, we should have $y' = f(x_1', \ldots, x_v')$.

16. Which Dependencies Are Scale Invariant

- For a single variable, if we plug in the expressions for y' and x' into the formula $y' = f(x')$, we get

$$\mu(\lambda) \cdot y = f(\lambda \cdot x).$$

- Here, $y = f(x)$, so $\mu(\lambda) \cdot f(x) = f(\lambda \cdot x)$.
- It is known that every measurable solution to this functional equation has the power law form $y = C \cdot x^m$.
- Similarly, for functions of several variables, we get

$$\mu(\lambda_1, \ldots, \lambda_v) \cdot f(x) = f(\lambda_1 \cdot x_1, \ldots, \lambda_v \cdot x_v).$$

- It is known that every measurable solution to this functional equation has the form

$$y = C \cdot x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}.$$

17. How Can We Use Scale Invariance Here?

- It would be nice to apply scale invariance to crack growth.
- However, we cannot directly use it – in the above arguments:
  - we assumed that y and x_i are different quantities, measured by different units,
  - but in our case Δa and a are both lengths.
- What can we do?
- To apply scale invariance, we can recall that in all applications, stress is periodic.
- For an engine, we know how many cycles per minute we have.
- For a road, we also know, on average, how many cars pass through the give road segment.

18. How Can We Use Scale Invariance (cont-d)

- In both cases, what we are really interested in is how much the crack will grow during some time interval.
- For example:
  - whether the road segment needs repairs right now
  - or it can wait until the next year.
- Thus, what we are really interested in is:
  - not the value Δa,
  - but the value $\dfrac{da}{dt}$ which can be obtained by multiplying Δa and # of cycles per unit time.

19. How Can We Use Scale Invariance (cont-d)

- The quantities $\frac{da}{dt}$ and Δa differ by a multiplicative constant.
- So, they follow the same laws as Δa.
- However, for $\frac{da}{dt}$, we already have different measuring units and thus, we can apply scale invariance.

20. So, Let Us Apply Scale Invariance

- For the case of one variable, scale invariance leads to the power law.
- This explains Wöhler law.
- For the case of several variables we similarly explain Paris law.
- Thus, both Wöhler and Paris laws can indeed be theoretically explained – by scale invariance.

21. Scale Invariance Explains How C Depends on m

- The coefficients C and m describing the Paris law are different for different materials.
- This means that, to determine how a specific crack will grow:
  - it is not sufficient to know its stress intensity K,
  - there must be some other characteristic z on which Δa depends: $\Delta a = f(K, z)$.
- Let us first apply scale invariance to the dependence of Δa on K.
- Then we can conclude that this dependence is described by a power law: $\Delta a(K, z) = C(z) \cdot K^{m(z)}$.
- In general, the coefficients $C(z)$ and $m(z)$ may depend on z.

22. How C Depends on m (cont-d)

- In log-log scale, we get a linear dependence:

$$\ln(\Delta a(K,z)) = m(z)\cdot \ln(K) + \ln(C(z)).$$

- If we apply scale invariance to the dependence of Δa on z, we get $\Delta a(K,z) = C'(K)\cdot z^{m'(K)}$ and:

$$\ln(\Delta a(K,z)) = m'(K)\cdot \ln(z) + \ln(C'(k)).$$

- So, $\ln(\Delta a(K,z))$ in linear in $\ln(K)$ and linear in $\ln(z)$.
- Thus it is a bilinear function of $\ln(K)$ and $\ln(z)$.
- A general bilinear function has the form:

$$\ln(\Delta a(K,z)) = a_0 + a_K\cdot\ln(K) + a_z\cdot\ln(z) + a_{Kz}\cdot\ln(K)\cdot\ln(z) =$$
$$(a_0 + a_z\cdot\ln(z)) + (a_K + a_{Kz}\cdot\ln(z))\cdot\ln(K).$$

- By applying $\exp(t)$ to both sides, we conclude that $\Delta a = C\cdot K^m$, where

$$C = \exp(a_0 + a_z\cdot\ln(z)) \text{ and } m = a_K + a_{Kz}\cdot\ln(z).$$

23. How C Depends on m (cont-d)

- So, $\ln(z) = \dfrac{1}{a_{Kz}} \cdot m - \dfrac{a_K}{a_{Kz}}$.
- Substituting this expression for $\ln(z)$ into the formula $C = \exp(a_0 + a_z \cdot \ln(z))$, we conclude that:

$$C = \exp\left(\left(a_0 - \frac{a_K \cdot a_z}{a_{Kz}}\right) + \frac{a_z}{a_{Kz}} \cdot m\right).$$

- Thus, $C = c_0 \cdot b_0^m$, with $c_0 = \exp\left(a_0 - \dfrac{a_K \cdot a_z}{a_{Kz}}\right)$ and $b_0 = \exp\left(\dfrac{a_z}{a_{Kz}}\right)$.
- Thus, the empirical dependence of C on m can also be explained by scale invariance.

24. Towards Explaining Generalized Paris Law

- So far, we have justified two laws:
  - Wöhler law that describes how cracks appear and start growing, and
  - Paris law that describes how they grow once they reach a certain size.
- In effect, these two laws describe two different mechanisms for crack growth.
- To describe the joint effect of these two mechanisms, we need to combine the effects of both mechanisms.

25. How Can We Combine the Two Formulas?

- Let us denote the effect of the i-th mechanism by q_i.
- A natural way to combine them is to consider some function $q = F(q_1, q_2)$.
- What should be the properties of this combination function?
- If one the effects is missing, then the overall effect should coincide with the other effect.
- So we should have $F(0, q_2) = q_2$ and $F(q_1, 0) = q_1$ for all q_1 and q_2.
- If we combine two effects, it should not matter in what order we consider them, i.e., we should have

$$F(q_1, q_2) = F(q_2, q_1) \text{ for all } q_1 \text{ and } q_2.$$

- In mathematical terms, the combination operation $F(q_1, q_2)$ should be *commutative*.

26. How to Combine the Formulas (cont-d)

- Similarly, if we combine three effects, the result should not depend on the order in which we combine them:

$$F(F(q_1, q_1), q_3) = F(q_1, F(q_2, q_3)) \text{ for all } q_1, q_1, \text{ and } q_3.$$

- In mathematical terms, the combination operation $F(q_1, q_2)$ should be *associative*.
- It is also reasonable to require that if we increase one of the effects, then the overall effect will increase.
- So, the function $F(q_1, q_2)$ should be *strictly monotonic* in each of the variables: if $q_1 < q_1'$, then we should have

$$F(q_1, q_2) < F(q_1', q_2).$$

27. How to Combine the Formulas (cont-d)

- It is also reasonable to require that small changes to q_i should lead to small changes in the overall effect.
- So, the function $F(q_1, q_2)$ should be *continuous*.
- Finally, it is reasonable to require that the operation $F(q_1, q_2)$ be *scale invariant* in the following sense:
  - if $q = F(q_1, q_2)$,
  - then for every $\lambda > 0$, for $q_i' = \lambda \cdot q_i$ and $q' = \lambda \cdot q$, we should have $q' = F(q_1', q_2')$.

28. This Explains the Generalized Paris Law

- It is known that every combination operation for which $F(q_1, 0) = q_1$:
  - if it is commutative, associative, strictly monotonic, continuous, and scale invariant,
  - then it has the form:

$$F(q_1, q_2) = (q_1^p + q_2^p)^{1/p} \text{ for some } p > 0.$$

- For $q_1 = C_0 \cdot \sigma^{m_0}$ and $q_2 = C \cdot \sigma^m \cdot a^{m'}$, we get:

$$\Delta a = \left((C_0 \cdot \sigma^{m_0})^p + \left(C \cdot \sigma^m \cdot a^{m'}\right)^p\right)^{1/p} =$$

$$\left(C_0^p \cdot \sigma^{m_0 \cdot p} + C^p \cdot \sigma^{m \cdot p} \cdot a^{m' \cdot p}\right)^{1/p} =$$

$$C \cdot \sigma^m \cdot \left(a^{m' \cdot p} + \left(\frac{C_0}{C}\right)^p \cdot \sigma^{(m - m_0) \cdot p}\right)^{1/p}.$$

29. Generalized Paris Law (cont-d)

- Reminder:

$$\left(C_0^p \cdot \sigma^{m_0 \cdot p} + C^p \cdot \sigma^{m \cdot p} \cdot a^{m' \cdot p}\right)^{1/p} =$$

$$C \cdot \sigma^m \cdot \left(a^{m' \cdot p} + \left(\frac{C_0}{C}\right)^p \cdot \sigma^{(m-m_0)\cdot p}\right)^{1/p}.$$

- This is exactly the generalized Paris Law

$$\Delta a = C \cdot \sigma^m \cdot \left(a^\alpha + c \cdot \sigma^\beta\right)^\gamma, \text{ with}$$

$$\alpha = m' \cdot p, c = \left(\frac{C_0}{C}\right)^p, \beta = (m - m_0) \cdot p, \text{ and } \gamma = 1/p.$$

- Thus, the generalized Paris law can also be explained by scale invariance.

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