How Transition from Purely Constructive Mathematics to Physics-Motivated Intuitionistic Mathematics Affects Decidability: An Important Facet of Mints's Legacy

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1. Constructive Mathematics

- Many processes from the physical world are described by mathematical equations.
- Traditional (non-constructive) mathematics can help us prove the existence of a solution to given the equations.
- However, existence proofs are often *non-constructive*: they do not help us compute the solution.
- Moreover, in traditional mathematics, it is not easy even to describe the existence of an algorithm.
- So logicians invented *constructive mathematics*, where $\exists x$ means that we have an algorithm for constructing x.
- Then, the not-necessarily-constructive existence is described, e.g., by $\neg\neg\exists x$.

2. Beyond Constructive Mathematics

- In constructive mathematics, only constructive objects are possible.
- For applications, this is a serious limitation: non-computable objects are possible.
- For example, data may come come from a random process – like quantum measurement.
- This limitation was one of the main motivations for G. Mints to consider:
  - a more general intuitionistic-style constructive mathematics,
  - where non-computable objects are allowed.
- In this talk, we study the relation between physics and the corresponding version of constructive mathematics.

Part I

Taking Into Account that We Process Physical Data

3. Need to Supplement Probabilistic Information with Information re What Is Possible

- Physical laws enable us to predict probabilities p.
- In general, probability p is a frequency f with which an event occurs, but sometimes, $f \neq p$.
- Example: due to molecular motion, a cold kettle on a cold stove can spontaneously boil with $p > 0$.
- However, most physicists believe that this event is simply not possible.
- This impossibility cannot be described by claiming that for some p_0, events with $p \leq p_0$ are not possible.
- Indeed, if we toss a coin many times N, we can get $2^{-N} < p_0$, but the result is still possible.
- So, to describe physics, we need to supplement probabilities with information on what is possible.

4. How to Describe What Is Possible

- Let U be the set of possible events.
- We assume that we know the probabilities $p(S)$ of different events $S \subseteq U$.
- From all possible events, the expert select a subset T of all events which are possible.
- The main idea that if the probability is very small, then the corresponding event is not possible.
- What is "very small" depends on the situation.
- Let $A_1 \supseteq A_2 \supseteq \ldots \supset A_n \supseteq \ldots$ be a definable sequence of events with $p(A_n) \to 0$.
- Then for some sufficiently large N, the probability of the corresponding event A_N becomes very small.
- Thus, the event A_N is not impossible, i.e., $T \cap A_N = \emptyset$.

5. Resulting Definitions

- Let U be a set with a probability measure p.
- We say that $T \subseteq U$ is *a set of possible elements* if:
    - for every definable sequence A_n for which $A_n \supseteq A_{n+1}$ and $p(A_n) \to 0$,
    - there exists N for which $T \cap A_N = \emptyset$.
- Physicists uses a similar argument even when do not know probabilities.
- For example, they usually claim that:
    - when x is small,
    - quadratic terms in Taylor expansion $a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots$ can be safely ignored.
- Theoretically, we can have a_2 s.t. $|a_2 \cdot x^2| \gg |a_1 \cdot x|$.
- However, physicists believe that such a_2 are not physically possible.

6. Definitions (cont-d)

- Physicists believe that very large values of a_2 are not physically possible.
- Here, we have $A_n = \{a_2 : |a_2| \geq n\}$.
- The physicists' belief is that for a sufficiently large N, event A_N is impossible, i.e., $A_N \cap T = \emptyset$.
- Here, $\cap A_n = \emptyset$, so $p(A_n) \to 0$ for any probability measure p.
- There are other similar conclusions, so we arrive at the following definition.
- We say that $T \subseteq U$ is *a set of possible elements* if:
  - for every definable sequence A_n for which $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$,
  - there exists N for which $T \cap A_N = \emptyset$.

7. In General, Many Problems Are Not Algorithmically Decidable

- A simple example is that it is impossible to decide whether two computable real numbers are equal or not.
- What are computable real numbers?
- In practice, real numbers come from measurements, and measurements are never absolutely accurate.
- In principle, we can measure a real number x with higher and higher accuracy.
- For any n, we can measure x with accuracy 2^{-n}, and get a rational r_n for which $|x - r_n| \le 2^{-n}$.
- A real number is called computable if there is a procedure that, given n, returns x_n.

8. Many Problems Are Not Algorithmically Decidable (cont-d)

- Computing with computable real numbers means that,
  - in addition to usual computational steps,
  - we can also, given n, ask for r_n.
- Some things can be computed: e.g., given x and y, we can compute $z = x + y$.
- However, it is not possible to algorithmically check whether $x = y$.
- Indeed, suppose that this was possible.
- Then, for $x = y = 0$ with $r_n = s_n = 0$ for all n, our procedure will return "yes".
- This procedure consists of finitely many steps, thus it can only ask for finitely many values r_n and s_n.

9. Many Problems Are Not Algorithmically Decidable (cont-d)

- The $x \stackrel{?}{=} y$ procedure consists of finitely many steps, thus it can only ask for finitely many values r_n and s_n.
- Let N be the smallest number which is larger than all such requests n. So:
  - if we keep $x = 0$ and take $y' = 2^{-N} \neq 0$ with $s'_1 = \ldots = s'_{N-1} = 0$ and $s'_N = s'_{N+1} = \ldots = 2^{-N}$,
  - our procedure will not notice the difference and mistakenly return "yes".
- This proves that a procedure for checking whether two computable numbers are equal is not possible.
- Similar negative results are known for many other problems.

10. Under Possibility Information, Equality Becomes Decidable: Known Result

- On the set $U = \mathbb{R} \times \mathbb{R}$ of all possible pairs of real numbers, we have a subset T of possible numbers.
- In particular, we can consider the following definable sequence of sets $A_n \stackrel{\text{def}}{=} \{(x, y) : 0 < |x - y| \le 2^{-n}\}$.
- One can easily see that $A_n \supseteq A_{n+1}$ for all n and that $\cap A_n = \emptyset$.
- Thus, there exists a natural number N for which no element $s \in T$ belongs to the set A_N.
- This, in turn, means that for every pair $(x, y) \in T$, either $|x - y| = 0$ (i.e., $x = y$) or $|x - y| > 2^{-N}$.
- So, to check whether $x = y$ or not, it is sufficient to compute both x and y with accuracy $2^{-(N+2)}$.

11. Under Possibility Information, Many Problems Become Decidable: A New Result

- In terms of sequences r_n and s_n, equality $x = y$ can be described as $\forall n\,(|r_n - s_n| \le 2^{-(n-1)})$.
- Many properties involving limits, differentiability, etc., can be described by *arithmetic formulas*

$$\Phi \stackrel{\text{def}}{=} Qn_1\, Qn_2 \ldots Qn_k\, F(r_1, \ldots, r_\ell, n_1, \ldots, n_k).$$

- Here, Qn_i is $\forall n_i$ or $\exists n_i$; $r_1, \ldots, r_\ell$ are sequences.
- F is a propositional combination of $=$'s and $\neq$'s between computable rational-valued expressions.
- For every Φ, for every set T of possible tuples $r = (r_1, \ldots, r_\ell)$, there exists an algorithm that,
  - given a tuple $r = (r_1, \ldots, r_\ell) \in T$,
  - checks whether Φ is true.

12. Proof by Quantifier Elimination

- We show that an expression $\exists n_i\, G(n_i)$ or $\forall n_i\, G(n_i)$ is equivalent to a quantifier-free formula.
- Here, $\exists n_i\, G(n_i) \Leftrightarrow \neg\forall n_i\, \neg G(n_i)$, so it is sufficient to prove it for $\forall$.
- Then, by eliminating quantifiers one by one, we get an equivalent easy-to-check quantifier-free formula.
- Take $A_n = \{r : \forall n_1\, (n_1 \le n \to G(n_1))\, \&\, \neg\forall n_1\, G(n_1)\}$.
- One can easily check that $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$.
- Thus, there exists N for which $T \cap A_N = \emptyset$.
- So, for $r \in T$, if $\forall n_1\, (n_1 \le N \to G(n_1))$, we cannot have $\neg\forall n_1\, G(n_1)$, so we must have $\forall n_1\, G(n_1)$.
- Thus, for $r \in T$, $\forall n_1\, G(n_1)$ is equivalent to a quantifier-free formula $G(1)\, \&\, G(2)\, \&\, \ldots\, \&\, G(N)$.

Part II

How to Take into Account that We Can Use Non-Standard Physical Phenomena to Process Data

13. Solving NP-Complete Problems Is Important

- In practice, we often need to find a solution that satisfies a given set of constraints.
- At a minimum, we need to check whether such a solution is possible.
- Once we have a candidate, we can feasibly check whether this candidate satisfies all the constraints.
- In theoretical computer science, "feasibly" is usually interpreted as computable in polynomial time.
- The class of all such problems is called NP.
- Example: satisfiability – checking whether a formula like $(v_1 \vee \neg v_2 \vee v_3) \,\&\, (v_4 \vee \neg v_2 \vee \neg v_5) \,\&\, \ldots$ can be true.
- Each problem from the class NP can be algorithmically solved by trying all possible candidates.

14. NP-Complete Problems (cont-d)

- For example, we can try all 2^n possible combinations of true-or-false values $v_1, \ldots, v_n$.
- For medium-size inputs, e.g., for $n \approx 300$, the resulting time 2^n is larger than the lifetime of the Universe.
- So, these exhaustive search algorithms are not practically feasible.
- It is not known whether problems from the class NP can be solved feasibly (i.e., in polynomial time).
- This is the famous open problem $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$.
- We know that some problems are *NP-complete*: every problem from NP can be reduced to it.
- So, it is very important to be able to efficiently solve even one NP-hard problem.

15. Can Non-Standard Physics Speed Up the Solution of NP-Complete Problems?

- NP-complete means difficult to solve on computers based on the usual physical techniques.
- A natural question is: can the use of non-standard physics speed up the solution of these problems?
- This question has been analyzed for several specific physical theories, e.g.:
  - for quantum field theory,
  - for cosmological solutions with wormholes and/or casual anomalies.
- So, a scheme based on a theory may not work.

16. No Physical Theory Is Perfect

- If a speed-up is possible within a given theory, is this a satisfactory answer?
- In the history of physics,
  - always new observations appear
  - which are not fully consistent with the original theory.
- For example, Newton's physics was replaced by quantum and relativistic theories.
- Many physicists believe that every physical theory is approximate.
- For each theory T, inevitably new observations will surface which require a modification of T.
- Let us analyze how this idea affects computations.

17. No Physical Theory Is Perfect: How to Formalize This Idea

- *Statement:* for every theory, eventually there will be observations which violate this theory.
- To formalize this statement, we need to formalize what are *observations* and what is a *theory*.
- Most sensors already produce *observation* in the computer-readable form, as a sequence of 0s and 1s.
- Let ω_i be the bit result of an experiment whose description is i.
- Thus, all past and future observations form a (potentially) infinite sequence $\omega = \omega_1\omega_2\ldots$ of 0s and 1s.
- A physical *theory* may be very complex.
- All we care about is which sequences of observations ω are consistent with this theory and which are not.

18. What Is a Physical Theory?

- So, a physical theory T can be defined as the set of all sequences ω which are consistent with this theory.
- A physical theory must have at least one possible sequence of observations: $T \neq \emptyset$.
- A theory must be described by a finite sequence of symbols: the set T must be *definable*.
- How can we check that an infinite sequence $\omega = \omega_1\omega_2\ldots$ is consistent with the theory?
- The only way is check that for every n, the sequence $\omega_1 \ldots \omega_n$ is consistent with T; so:

$$\forall n\, \exists \omega^{(n)} \in T\, (\omega_1^{(n)} \ldots \omega_n^{(n)} = \omega_1 \ldots \omega_n) \Rightarrow \omega \in T.$$

- In mathematical terms, this means that T is *closed* in the Baire metric $d(\omega, \omega') \stackrel{\text{def}}{=} 2^{-N(\omega,\omega')}$, where

$$N(\omega, \omega') \stackrel{\text{def}}{=} \max\{k : \omega_1 \ldots \omega_k = \omega'_1 \ldots \omega'_k\}.$$

19. What Is a Physical Theory: Definition

- A theory must predict something new.
- So, for every sequence $\omega_1 \ldots \omega_n$ consistent with T, there is a continuation which does not belong to T.
- In mathematical terms, T is *nowhere dense.*
- *By a* physical theory, *we mean a non-empty closed nowhere dense definable set* T*.*
- *A sequence* ω *is* consistent with the no-perfect-theory principle *if it does not belong to any physical theory.*
- In precise terms, ω does not belong to the union of all definable closed nowhere dense set.
- There are countably many definable set, so this union is *meager* (= *Baire first category*).
- Thus, due to Baire Theorem, such sequences ω exist.

20. How to Represent Instances of an NP-Complete Problem

- For each NP-complete problem $\mathcal{P}$, its instances are sequences of symbols.
- In the computer, each such sequence is represented as a sequence of 0s and 1s.
- We can append 1 in front and interpret this sequence as a binary code of a natural number i.
- In principle, not all natural numbers i correspond to instances of a problem $\mathcal{P}$.
- We will denote the set of all natural numbers which correspond to such instances by $S_{\mathcal{P}}$.
- For each $i \in S_{\mathcal{P}}$, we denote the correct answer (true or false) to the i-th instance of the problem $\mathcal{P}$ by $s_{\mathcal{P},i}$.

21. What We Mean by Using Physical Observations in Computations

- In addition to performing computations, our computational device can:
  - produce a scheme i for an experiment, and then
  - use the result ω_i of this experiment in future computations.
- In other words, given an integer i, we can produce ω_i.
- In precise terms, the use of physical observations in computations means that use ω as an *oracle*.

22. Main Result

- A *ph-algorithm* $\mathcal{A}$ is an algorithm that uses an oracle ω consistent with the no-perfect-theory principle.
- The result of applying an algorithm $\mathcal{A}$ using ω to an input i will be denoted by $\mathcal{A}(\omega, i)$.
- We say that a feasible ph-algorithm $\mathcal{A}$ *solves almost all instances of an NP-complete problem* $\mathcal{P}$ if:

$$\forall \varepsilon_{>0}\, \forall n\, \exists N_{\ge n} \left(\frac{\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \le N : i \in S_{\mathcal{P}}\}} > 1 - \varepsilon \right).$$

- Restriction to sufficiently long inputs $N \ge n$ makes sense: for short inputs, we can do exhaustive search.
- **Theorem.** *For every NP-complete problem* $\mathcal{P}$*, there is a feasible ph-alg.* $\mathcal{A}$ *solving almost all instances of* $\mathcal{P}$*.*

23. This Result Is the Best Possible

- Our result is the best possible, in the sense that the use of physical observations cannot solve *all* instances:
- **Proposition.** *If P≠NP, then no feasible ph-algorithm $\mathcal{A}$ can solve all instances of $\mathcal{P}$.*
- Can we prove the result for *all* N starting with some N_0?
- We say that a feasible ph-algorithm $\mathcal{A}$ *δ-solves* $\mathcal{P}$ if

$$\exists N_0\, \forall N \ge N_0 \left(\frac{\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \le N : i \in S_{\mathcal{P}}\}} > \delta \right).$$

- **Proposition.** *For every NP-complete problem $\mathcal{P}$ and for every $\delta > 0$:*
  - *if there exists a feasible ph-algorithm $\mathcal{A}$ that δ-solves $\mathcal{P}$,*
  - *then there is a feasible algorithm $\mathcal{A}'$ that also δ-solves $\mathcal{P}$.*

Part III

Physical and Computational Consequences

24. Finding Roots

- In general, it is not possible, given a f-n $f(x)$ attaining negative and positive values, to compute its root.
- This becomes possible if we restrict ourselves to physically meaningful functions:
- *Let K be a computable compact.*
- *Let X be the set of all functions $f : K \to \mathbb{R}$ that attain 0 value somewhere on K. Then:*
  - *for every set $\mathcal{T} \subseteq X$ consisting of physically meaningful functions and for every $\varepsilon > 0$,*
  - *there is an algorithm that, given a f-n $f \in \mathcal{T}$, computes an ε-approximation to the set of roots*

$$R \stackrel{\text{def}}{=} \{x : f(x) = 0\}.$$

- In particular, we can compute an ε-approximation to one of the roots.

25. Optimization

- In general, it is not algorithmically possible to find x where $f(x)$ attains maximum.
- Let K be a computable compact. Let X be the set of all functions $f : K \to \mathbb{R}$. Then:
  - *for every set $\mathcal{T} \subseteq X$ consisting of physically meaningful functions and for every $\varepsilon > 0$,*
  - *there is an algorithm that, given a f-n $f \in \mathcal{T}$, computes an ε-approx. to* $S = \left\{x : f(x) = \max\limits_y f(y)\right\}$.
- In particular, we can compute an approximation to an individual $x \in S$.
- *Reduction to roots:* $f(x) = \max\limits_y f(y)$ iff $g(x) = 0$, where $g(x) \stackrel{\text{def}}{=} f(x) - \max\limits_y f(y)$.

26. Computing Fixed Points

- In general, it is not possible to compute all the fixed points of a given computable function $f(x)$.
- Let K be a computable compact. Let X be the set of all functions $f : K \to K$. Then:
  - *for every set $\mathcal{T} \subseteq X$ consisting of physically meaningful functions and for every $\varepsilon > 0$,*
  - *there is an algorithm that, given a f-n $f \in \mathcal{T}$, computes an ε-approximation to the set $\{x : f(x) = x\}$.*
- In particular, we can compute an approximation to an individual fixed point.
- *Reduction to roots:* $f(x) = x$ iff $g(x) = 0$, where $g(x) \stackrel{\text{def}}{=} d(f(x), x)$.

27. Computing Limits

- *In general:* it is not algorithmically possible to find a limit $\lim a_n$ of a convergent computable sequence.
- Let K be a computable compact. Let X be the set of all convergent sequences $a = \{a_n\}$, $a_n \in K$. Then:
  - *for every set $\mathcal{T} \subseteq X$ consisting of physically meaningful functions and for every $\varepsilon > 0$,*
  - *there exists an algorithm that, given a sequence $a \in \mathcal{T}$, computes its limit with accuracy ε.*
- *Use:* this enables us to compute limits of iterations and sums of Taylor series (frequent in physics).
- *Main idea:* for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $|a_n - a_{n-1}| \le \delta$, then $|a_n - \lim a_n| \le \varepsilon$.
- *Intuitively:* we stop when two consequent iterations are close to each other.

28. Justification of Physical Induction

- *What is physical induction:* a property P is satisfied in the first N experiments, then it is satisfied always.
- *Comment:* N should be sufficiently large.
- *Theorem:* $\forall \mathcal{T}\,\exists N$ s.t. if for $o \in \mathcal{T}$, $P(o)$ is satisfied in the first N experiments, then $P(o)$ is satisfied always.
- *Notation:* $s \stackrel{\text{def}}{=} s_1 s_2 \ldots$, where:
  - $s_i = T$ if $P(o)$ holds in the i-th experiment, and
  - $s_i = F$ if $\neg P(o)$ holds in the i-th experiment.
- *Proof:* $A_n \stackrel{\text{def}}{=} \{o : s_1 = \ldots = s_n = T \,\&\exists m\,(s_m = F)\}$; then $A_n \supseteq A_{n+1}$ and $\cup A_n = \emptyset$ so $\exists N\,(A_N \cap \mathcal{T} = \emptyset)$.
- *Meaning* of $A_N \cap \mathcal{T} = \emptyset$: if $o \in \mathcal{T}$ and $s_1 = \ldots = s_N = T$, then $\neg\exists m\,(s_m = F)$, i.e., $\forall m\,(s_m = T)$.

29. Ill-Posted Problem: Brief Reminder

- Main *objectives* of science:
  - *guaranteed* estimates for physical quantities;
  - *guaranteed* predictions for these quantities.
- *Problem:* estimation and prediction are ill-posed.
- *Example:*
  - measurement devices are inertial;
  - hence suppress high frequencies ω;
  - so $\varphi(x)$ and $\varphi(x) + \sin(\omega \cdot t)$ are indistinguishable.
- *Existing approaches:*
  - statistical regularization (filtering);
  - Tikhonov regularization (e.g., $|\dot{x}| \le \Delta$);
  - expert-based regularization.
- *Main problem:* no guarantee.

30. On Physically Meaningful Solutions, Problems Become Well-Posed

- *State estimation – an ill-posed problem:*
  - *Measurement f:*
    state $s \in S \to$ observation $r = f(s) \in R$.
  - *In principle,* we can reconstruct $r \to s$:
    as $s = f^{-1}(r)$.
  - *Problem:* small changes in r can lead to huge changes in s (f^{-1} *not continuous*).
- *Theorem:*
  - Let S be a definably separable metric space.
  - Let $\mathcal{T}$ be a set of physically meaningful elements of S.
  - Let $f : S \to R$ be a continuous 1-1 function.
  - Then, the inverse mapping $f^{-1} : R \to S$
    is *continuous* for every $r \in f(\mathcal{T})$.

31. Einstein-Podolsky-Rosen (EPR) Paradox

- Due to *Relativity Theory*, two spatially separated simultaneous events cannot influence each other.
- *Einstein, Podolsky, and Rosen* intended to show that in quantum physics, such influence is possible.
- *In formal terms*, let x and x' be measured values at these two events.
- *Independence* means that possible values of x do not depend on x', i.e., $\mathcal{T} = X \times X'$ for some X and X'.
- *Physical induction* implies that the pair (x, x') belongs to a set S of physically meaningful pairs.
- **Theorem.** *A set $\mathcal{T}$ os physically meaningful pairs cannot be represented as $X \times X'$.*
- Thus, everything *is* related – but we probably can't use this relation to pass information ($\mathcal{T}$ isn't computable).

32. When to Stop an Iterative Algorithm?

- *Situation* in numerical mathematics:
  - we often know an iterative process whose results x_k are known to converge to the desired solution x,
  - but we do not know when to stop to guarantee that

$$d_X(x_k, x) \le \varepsilon.$$

- *Heuristic approach:* stop when $d_X(x_k, x_{k+1}) \le \delta$ for some $\delta > 0$.
- *Example:* in physics, if 2nd order terms are small, we use the linear expression as an approximation.

33. When to Stop an Iterative Algorithm: Result

- Let $\{x_k\} \in \mathcal{T}$, k be an integer, and $\varepsilon > 0$ a real number.
- We say that x_k is ε-*accurate* if $d_X(x_k, \lim x_p) \le \varepsilon$.
- Let $d \ge 1$ be an integer.
- By a *stopping criterion*, we mean a function $c : X^d \to R_0^+$ that satisfies the following two properties:
  - If $\{x_k\} \in \mathcal{T}$, then $c(x_k, \ldots, x_{k+d-1}) \to 0$.
  - If for some $\{x_n\} \in \mathcal{T}$ and k, $c(x_k, \ldots, x_{k+d-1}) = 0$, then $x_k = \ldots = x_{k+d-1} = \lim x_p$.
- *Result:* Let c be a stopping criterion. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that
  – if $c(x_k, \ldots, x_{k+d-1}) \le \delta$, and the sequence $\{x_n\}$ is physically meaningful,
  – then x_k is ε-accurate.

Part IV
Relation with Randomness

34. Towards Relation with Randomness

- If a sequence s is random, it satisfies all the probability laws such as the law of large numbers.
- If a sequence satisfies all probability laws, then for all practical purposes we can consider it random.
- Thus, we can define a sequence to be random if it satisfies all probability laws.
- A probability law is a statement S which is true with probability 1: $P(S) = 1$.
- So, a sequence is random if it belongs to all definable sets of measure 1.
- A sequence belongs to a set of measure 1 iff it does not belong to its complement $C = -S$ with $P(C) = 0$.
- So, *a sequence is* random *if it does not belong to any definable set of measure 0.*

35. Randomness and Kolmogorov Complexity

- Different definabilities lead to different randomness.
- When definable means computable, randomness can be described in terms of Kolmogorov complexity

$$K(x) \stackrel{\text{def}}{=} \min\{\text{len}(p) : p \text{ generates } x\}.$$

- Crudely speaking, an infinite string $s = s_1 s_2 \ldots$ is random if, for some constant $C > 0$, we have

$$\forall n\, (K(s_1 \ldots s_n) \geq n - C).$$

- Indeed, if a sequence $s_1 \ldots s_n$ is truly random, then the only way to generate it is to explicitly print it:

$$\texttt{print}(s_1 \ldots s_n).$$

- In contrast, a sequence like 0101...01 generated by a short program is clearly not random.

36. From Kolmogorov-Martin-Löf Theoretical Randomness to a More Physical One

- The above definition means that (definable) events with probability 0 cannot happen.
- In practice, physicists also assume that events with a *very small* probability cannot happen.
- For example, a kettle on a cold stove will not boil by itself – but the probability is non-zero.
- If a coin falls head 100 times in a row, any reasonable person will conclude that this coin is not fair.
- It is not possible to formalize this idea by simply setting a threshold $p_0 > 0$ below which events are not possible.
- Indeed, then, for N for which $2^{-N} < p_0$, no sequence of N heads or tails would be possible at all.

37. From Kolmogorov-Martin-Löf Theoretical Randomness to a More Physical One (cont-d)

- We cannot have a universal threshold p_0 such that events with probability $\leq p_0$ cannot happen.
- However, we know that:
  - for each decreasing ($A_n \supseteq A_{n+1}$) sequence of properties A_n with $\lim p(A_n) = 0$,
  - there exists an N above which a truly random sequence cannot belong to A_N.
- *Resulting definition:* we say that $\mathcal{R}$ is a *set of random elements* if
  - for every definable decreasing sequence $\{A_n\}$ for which $\lim P(A_n) = 0$,
  - there exists an N for which $\mathcal{R} \cap A_N = \emptyset$.

38. Random Sequences and Physically Meaningful Sequences

- Let $\mathcal{R}_K$ denote the set of all elements which are random in Kolmorogov-Martin-Löf sense. Then:
- *Every set of random elements consists of physically meaningful elements.*
- *For every set $\mathcal{T}$ of physically meaningful elements, the intersection $\mathcal{T} \cap \mathcal{R}_K$ is a set of random elements.*
- *Proof:* When A_n is definable, for $D_n \stackrel{\text{def}}{=} \bigcap\limits_{i=1}^{n} A_i - \bigcap\limits_{i=1}^{\infty} A_i$, we have $D_n \supseteq D_{n+1}$ and $\bigcap\limits_{n=1}^{\infty} D_n = \emptyset$, so $P(D_n) \to 0$.
- Therefore, there exists an N for which the set of random elements does not contain any elements from D_N.
- Thus, every set of random elements indeed consists of physically meaningful elements.

Part V
Proofs

39. A Formal Definition of Definable Sets

- Let $\mathcal{L}$ be a theory.
- Let $P(x)$ be a formula from $\mathcal{L}$ for which the set $\{x \mid P(x)\}$ exists.
- We will then call the set $\{x \mid P(x)\}$ $\mathcal{L}$-*definable*.
- Crudely speaking, a set is $\mathcal{L}$-definable if we can explicitly *define* it in $\mathcal{L}$.
- All usual sets are definable: $\mathbb{N}$, $\mathbb{R}$, etc.
- Not every set is $\mathcal{L}$-definable:
  - every $\mathcal{L}$-definable set is uniquely determined by a text $P(x)$ in the language of set theory;
  - there are only countably many texts and therefore, there are only countably many $\mathcal{L}$-definable sets;
  - so, some sets of natural numbers are not definable.

40. How to Prove Results About Definable Sets

- Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets. Therefore:
  - in addition to the theory $\mathcal{L}$,
  - we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a countable set.
- For every formula F from the theory $\mathcal{L}$, we denote its Gödel number by $\lfloor F \rfloor$.
- We say that a theory $\mathcal{M}$ is *stronger* than $\mathcal{L}$ if:
  - $\mathcal{M}$ contains all formulas, all axioms, and all deduction rules from $\mathcal{L}$, and
  - $\mathcal{M}$ contains a predicate $\text{def}(n, x)$ such that for every formula $P(x)$ from $\mathcal{L}$ with one free variable,

$$\mathcal{M} \vdash \forall y\, (\text{def}(\lfloor P(x) \rfloor, y) \leftrightarrow P(y)).$$

41. Existence of a Stronger Theory

- As $\mathcal{M}$, we take $\mathcal{L}$ plus all above equivalence formulas.
- Is $\mathcal{M}$ consistent?
- Due to compactness, we prove that for any $P_1(x), \ldots, P_m(x)$, $\mathcal{L}$ is consistent with the equivalences corr. to $P_i(x)$.
- Indeed, we can take

$$\mathrm{def}(n,y) \leftrightarrow (n = \lfloor P_1(x) \rfloor \,\&\, P_1(y)) \vee \ldots \vee (n = \lfloor P_m(x) \rfloor \,\&\, P_m(y)).$$

- This formula is definable in $\mathcal{L}$ and satisfies all m equivalence properties.
- Thus, the existence of a stronger theory is proven.
- The notion of an $\mathcal{L}$-definable set can be expressed in $\mathcal{M}$: S is $\mathcal{L}$-definable iff $\exists n \in \mathbb{N}\, \forall y\, (\mathrm{def}(n,y) \leftrightarrow y \in S)$.
- So, all statements involving definability become statements from the $\mathcal{M}$ itself, *not* from metalanguage.

42. Consistency Proof

- *Statement:* $\forall \varepsilon > 0$, there exists a set $\mathcal{T}$ for which $\underline{P}(\mathcal{T}) \geq 1 - \varepsilon$.
- There are countably many definable sequences $\{A_n\}$: $\{A_n^{(1)}\}$, $\{A_n^{(2)}\}$, ...
- For each k, $P\left(A_n^{(k)}\right) \to 0$ as $n \to \infty$.
- Hence, there exists N_k for which $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$.
- We take $\mathcal{T} \stackrel{\text{def}}{=} - \bigcup\limits_{k=1}^{\infty} A_{N_k}^{(k)}$. Since $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$, we have
$$\overline{P}\left(\bigcup_{k=1}^{\infty} A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} P\left(A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} \varepsilon \cdot 2^{-k} = \varepsilon.$$
- Hence, $\underline{P}(\mathcal{T}) = 1 - \overline{P}\left(\bigcup\limits_{k=1}^{\infty} A_{N_k}^{(k)}\right) \geq 1 - \varepsilon$.

43. Finding Roots: Proof

- To compute the set $R = \{x : f(x) = 0\}$ with accuracy $\varepsilon > 0$, let us take an $(\varepsilon/2)$-net $\{x_1, \ldots, x_n\} \subseteq K$.
- For each i, we can compute $\varepsilon' \in (\varepsilon/2, \varepsilon)$ for which $B_i \stackrel{\text{def}}{=} \{x : d(x, x_i) \le \varepsilon'\}$ is a computable compact set.
- It is possible to algorithmically compute the minimum of a function on a computable compact set.
- Thus, we can compute $m_i \stackrel{\text{def}}{=} \min\{|f(x)| : x \in B_i\}$.
- Since $f \in T$, similarly to the previous proof, we can prove that $\exists N \, \forall f \in T \, \forall i \, (m_i = 0 \vee m_i \ge 2^{-N})$.
- Comp. m_i w/acc. $2^{-(N+2)}$, we check $m_i = 0$ or $m_i > 0$.
- Let's prove that $d_H(R, \{x_i : m_i = 0\}) \le \varepsilon$, i.e., that $\forall i \, (m_i = 0 \Rightarrow \exists x \, (f(x) = 0 \,\&\, d(x, x_i) \le \varepsilon))$ and $\forall x \, (f(x) = 0 \Rightarrow \exists i \, (m_i = 0 \,\&\, d(x, x_i) \le \varepsilon))$.

44. Finding Roots: Proof (cont-d)

- $m_i = 0$ means $\min\{|f(x)| : x \in B_i \stackrel{\text{def}}{=} B_{\varepsilon'}(x_i)\} = 0$.
- Since the set K is compact, this value 0 is attained, i.e., there exists a value $x \in B_i$ for which $f(x) = 0$.
- From $x \in B_i$, we conclude that $d(x, x_i) \le \varepsilon'$ and, since $\varepsilon' < \varepsilon$, that $d(x, x_i) < \varepsilon$.
- Thus, x_i is ε-close to the root x.
- Vice versa, let x be a root, i.e., let $f(x) = 0$.
- Since the points x_i form an $(\varepsilon/2)$-net, there exists an index i for which $d(x, x_i) \le \varepsilon/2$.
- Since $\varepsilon/2 < \varepsilon'$, this means that $d(x, x_i) \le \varepsilon'$ and thus, $x \in B_i$.
- Therefore, $m_i = \min\{|f(x)| : x \in B_i\} = 0$. So, the root x is ε-close to a point x_i for which $m_i = 0$.

45. Proof of Well-Posedness

- *Known:* if a f is continuous and 1-1 on a compact, then f^{-1} is also continuous.
- *Reminder:* S is compact if and only if it is closed and for every ε, it has a finite ε-net.
- *Given:* the set X is definably separable.
- *Means:* $\exists$ def. $s_1, \ldots, s_n, \ldots$ everywhere dense in X.
- *Solution:* take $A_n \stackrel{\text{def}}{=} -\bigcup\limits_{i=1}^{n} B_\varepsilon(s_i)$.
- Since s_i are everywhere dense, we have $\cap A_n = \emptyset$.
- Hence, there exists N for which $A_N \cap \mathcal{T} = \emptyset$.
- Since $A_N = -\bigcup\limits_{i=1}^{N} B_\varepsilon(s_i)$, this means $\mathcal{T} \subseteq \bigcup\limits_{i=1}^{N} B_\varepsilon(s_i)$.
- Hence $\{s_1, \ldots, s_N\}$ is an ε-net for $\mathcal{T}$. Q.E.D.

46. Random Sequences and Physically Meaningful Sequences (proof cont-d)

- Let T consist of physically meaningful elements. Let us prove that $\mathcal{T} \cap \mathcal{R}_K$ is a set of random elements.
- If $A_n \supseteq A_{n+1}$ and $P\left(\bigcap\limits_{n=1}^{\infty} A_n\right) = 0$, then for $B_m \stackrel{\text{def}}{=} A_m - \bigcap\limits_{n=1}^{\infty} A_n$, we have $B_m \supseteq B_{m+1}$ and $\bigcap\limits_{n=1}^{\infty} B_n = \emptyset$.
- Thus, by definition of a set consisting of physically meaningful elements, we conclude that $B_N \cap T = \emptyset$.
- Since $P\left(\bigcap\limits_{n=1}^{\infty} A_n\right) = 0$, we also know that $\left(\bigcap\limits_{n=1}^{\infty} A_n\right) \cap \mathcal{R}_K = \emptyset$.
- Thus, $A_N = B_N \cup \left(\bigcap\limits_{n=1}^{\infty} A_n\right)$ has no common elements with the intersection $T \cap \mathcal{R}_K$. Q.E.D.

47. Using Non-Standard Physics: Proof of the Main Result

- As $\mathcal{A}$, given an instance i, we simply produce the result ω_i of the i-th experiment.
- Let us prove, by contradiction, that for every $\varepsilon > 0$ and for every n, there exists an integer $N \ge n$ for which

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} > (1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\}.$$

- The assumption that this property is not satisfied means that for some $\varepsilon > 0$ and for some integer n, we have

$$\forall N_{\ge n}\,\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} \le (1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\}.$$

- Let $T \stackrel{\text{def}}{=} \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le$

$$(1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge n\}.$$

- We will prove that this set T is a physical theory (in the sense of the above definition); then $\omega \notin T$.

48. Proof (cont-d)

- *Reminder:* $T = \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le$
$$(1-\varepsilon)\cdot\#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge n\}.$$
- By definition, a physical theory is a set which is non-empty, closed, nowhere dense, and definable.
- Non-emptiness is easy: the sequence $x_i = \neg s_{\mathcal{P},i}$ for $i \in S_{\mathcal{P}}$ belongs to T.
- One can prove that T is closed, i.e., if $x^{(m)} \in T$ for which $x^{(m)} \to \omega$, then $x \in T$.
- Nowhere dense means that for every finite sequence $x_1 \ldots x_m$, there exists a continuation $x \notin T$.
- Indeed, for extension, we can take $x_i = s_{\mathcal{P},i}$ if $i \in S_{\mathcal{P}}$.
- Finally, we have an explicit definition of T, so T is definable.

49. Non-Standard Physics: Proof of First Proposition

- Let us assume that P$\neq$NP; we want to prove that for every feasible ph-algorithm $\mathcal{A}$, it is not possible to have

$$\forall N\,(\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}}\}).$$

- Let us consider, for each feasible ph-algorithm $\mathcal{A}$,

$$T(\mathcal{A}) \stackrel{\text{def}}{=} \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(x, i) = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N\}.$$

- Similarly to the proof of the main result, we can show that this set $T(\mathcal{A})$ is closed and definable.
- To prove that $T(\mathcal{A})$ is nowhere dense, we extend $x_1 \ldots x_m$ by 0s; then $x \in T$ would mean P=NP.
- If $T(\mathcal{A}) \neq \emptyset$, then $T(\mathcal{A})$ is a theory, so $\omega \notin T(\mathcal{A})$.
- If $T(\mathcal{A}) = \emptyset$, this also means that $\mathcal{A}$ does not solve all instances of the problem $\mathcal{P}$ – no matter what ω we use.

50. Proof of Second Proposition

- Let us assume that no non-oracle feasible algorithm δ-solves the problem $\mathcal{P}$.
- Let's consider, for each N_0 and feasible ph-alg. $\mathcal{A}$,

$$T(\mathcal{A}, N_0) \stackrel{\text{def}}{=} \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(x, i) = s_{\mathcal{P},i}\} >$$
$$\delta \cdot \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge N_0\}.$$

- We want to prove that $\forall N_0\,(\omega \notin T(\mathcal{A}, N_0))$.
- Similarly to the proof of the Main Result, we can show that $T(\mathcal{A}, N_0)$ is closed and definable.
- To prove that $T(\mathcal{A}, N_0)$ is nowhere dense, we extend $x_1 \ldots x_m$ by 0s.
- If $T(\mathcal{A}, N_0) \ne \emptyset$, then $T(\mathcal{A}, N_0)$ is a theory hence $\omega \notin T(\mathcal{A}, N_0)$.
- If $T(\mathcal{A}, N_0) = \emptyset$, then also $\omega \notin T(\mathcal{A}, N_0)$.

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