Use of Maxitive (Possibility) Measures in Foundations of Physics and Description of Randomness: Case Study

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1. Physicists Assume that Initial Conditions and Values of Parameters are Not Abnormal

- To a mathematician, the main contents of a physical theory is its equations.
- Not all solutions of the equations have physical sense.
- *Ex. 1:* Brownian motion comes in one direction;
- *Ex. 2:* implosion glues shattered pieces into a statue;
- *Ex. 3:* fair coin falls heads 100 times in a row.
- *Mathematics:* it is possible.
- *Physics* (and common sense): it is not possible.
- *Our objective:* supplement probabilities with a new formalism that more accurately captures the physicists' reasoning.

2. A Seemingly Natural Formalizations of This Idea

- *Physicists:* only "not abnormal" situations are possible.
- *Natural formalization: idea.* If a probability $p(E)$ of an event E is small enough, then this event cannot happen.
- *Natural formalization: details.* There exists the "smallest possible probability" p_0 such that:
  - if the computed probability p of some event is larger than p_0, then this event can occur, while
  - if the computed probability p is $\le p_0$, the event cannot occur.
- *Example:* a fair coin falls heads 100 times with prob. 2^{-100}; it is impossible if $p_0 \ge 2^{-100}$.

3. The Above Formalization of the Notion of "Typical" is Not Always Adequate

- *Problem:* every sequence of heads and tails has exactly the same probability.
- *Corollary:* if we choose $p_0 \geq 2^{-100}$, we will thus exclude all sequences of 100 heads and tails.
- However, anyone can toss a coin 100 times, and this proves that some such sequences are physically possible.
- *Similar situation:* Kyburg's lottery paradox:
  - in a big (e.g., state-wide) lottery, the probability of winning the Grand Prize is so small that a reasonable person should not expect it;
  - however, some people do win big prizes.

4. Relation to Non-Monotonic Reasoning

- Traditional logic is *monotonic:* once a statement is derived it remains true.
- Expert reasoning is *non-monotonic:*
  - birds normally fly,
  - so, if we know only that Sam is a bird, we conclude that Sam flies;
  - however, if we learn the new knowledge that Sam is a penguin, we conclude that Sam doesn't fly.
- Non-monotonic reasoning helps resolve the lottery paradox (Poole et al.)
- *Our approach:* in fact, what we propose can be viewed as a specific non-monotonic formalism for describing rare events.

5. Events with 0 Probabilities are Possible: Another Explanation for the Lottery Paradox

- *Idea:* common sense intuition is false, events with small (even 0) probability are possible.
- This idea is promoted by known specialists in foundations of probability: K. Popper, B. De Finetti, G. Coletti, A. Gilio, R. Scozzafava, W. Spohn, etc.
- *Out attitude:* our objective is to formalize intuition, not to reject it.
- *Interesting:* both this approach and our approach lead to the same formalism (of maxitive measures).
- *Conclusion:* Maybe there is a deep relation and similarity between the two approaches.

6. New Idea

- *Example:* height:
  - if height is ≥ 6 ft, it is still normal;
  - if instead of 6 ft, we consider 6 ft 1 in, 6 ft 2 in, etc., then $\exists h_0$ s.t. everyone taller than h_0 is abnormal;
  - we are not sure what is h_0, but we are sure such h_0 exists.
- *General description:* on the universal set U, we have sets $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ s.t. $\bigcap\limits_n A_n = \emptyset$.
- *Example:* A_1 = people w/height ≥ 6 ft, A_2 = people w/height ≥ 6 ft 1 in, etc.
- A set $T \subseteq U$ is called a *set of typical (not abnormal) elements* if for every definable sequence of sets A_n for which $A_n \supseteq A_{n+1}$ for all n and $\bigcap\limits_n A_n = \emptyset$, there exists an integer N for which $A_N \cap T = \emptyset$.

7. Coin Example

- Universal set $U = \{\mathrm{H}, \mathrm{T}\}^{\mathbb{N}}$
- Here, A_n is the set of all the sequences that start with n heads and have at least one tail.
- The sequence $\{A_n\}$ is decreasing and definable, and its intersection is empty.
- Therefore, for every set T of typical elements of U, there exists an integer N for which $A_N \cap T = \emptyset$.
- This means that if a sequence $s \in T$ is not abnormal and starts with N heads, it must consist of heads only.
- In physical terms, it means a random sequence (i.e., a sequence that contains both heads and tails) cannot start with N heads.
- This is exactly what we wanted to formalize.

8. Main Result: Relation to Possibility Measures

- *Idea:* to describe a set of typical elements, we ascribe, to each definable monotonic sequence $\{A_n\}$, the smallest integer $N(\{A_n\})$ for which

$$A_N \cap T = \emptyset.$$

- This integer can be viewed as measure of *complexity* of the sequence:
  - for simple sequences, it is smaller,
  - for more complex sequences, it is larger.
- *In terms of complexity:* an element $x \in U$ is typical if and only if for every definable decreasing sequence $\{A_n\}$ with an empty intersection, $x \notin A_N$, where $N = N(\{A_n\})$ is the complexity of this sequence.
- *Theorem:* $N(\{A_n\})$ is a maxitive (possibility) measure, i.e., $N(\{A_n \cup B_n\}) = \max(N(\{A_n\}), N(\{B_n\}))$.

9. Possible Practical Use of This Idea: When to Stop an Iterative Algorithm

- *Situation* in numerical mathematics:
  - we often know an iterative process whose results x_k are known to converge to the desired solution x, but
  - we do not know when to stop to guarantee that

$$d_X(x_k, x) \le \varepsilon.$$

- *Heuristic approach:* stop when $d_X(x_k, x_{k+1}) \le \delta$ for some $\delta > 0$.
- *Example:* in physics, if 2nd order terms are small, we use the linear expression as an approximation.

10. Result

- *Definition.*
  - Let $\{x_k\} \in S$, k be an integer, and $\varepsilon > 0$ a real number. We say that x_k is $\varepsilon-$*accurate* if $d_X(x_k, \lim x_p) \le \varepsilon$.
  - Let $d \ge 1$ be an integer. By a *stopping criterion*, we mean a function $c : X^d \to R_0^+ = \{x \in R \,|\, x \ge 0\}$ that satisfies the following two properties:
    - If $\{x_k\} \in S$, then $c(x_k, \ldots, x_{k+d-1}) \to 0$.
    - If for some $\{x_n\} \in S$ and for some k, $c(x_k, \ldots, x_{k+d-1}) = 0$, then $x_k = \ldots = x_{k+d-1} = \lim x_p$.
- *Result:* Let c be a stopping criterion. Then, for every ε, there exists a $\delta > 0$ such that if a sequence $\{x_n\}$ is not abnormal, and $c(x_k, \ldots, x_{k+d-1}) \le \delta$, then x_k is $\varepsilon-$accurate.

11. Degree of Typicalness

- *Idea:* an atypical object may be a "typical" element of the class of abnormal objects, "typical exception".
- *Question:* how can we describe this?
- We start with the original theory $\mathcal{L}$.
- We then build a larger (meta-)theory $\mathcal{M}$, we can talk about definable and typical objects, about $T(A) \subseteq A$.
- To fully describe physicists' reasoning, we select a set $T(A)$ for all definable A.
- The original theory + selection forms a new theory $\mathcal{L}'$ in which, e.g., $T(A)$ is defined.
- On top of $\mathcal{L}'$, we can build a new meta-theory $\mathcal{M}'$.
- In $\mathcal{M}'$, we can talk about atypical elements of $Ab(A) \stackrel{\text{def}}{=} A \setminus T(A)$ – atypical of order 2, etc.

12. Beyond Maxitive Measures

- *Objective:* exclude events A of low probability.
- The threshold probability $c(A)$ should depend on the complexity of the event A.
- *Definition:* x is "reasonable" if for every probability measure p, there is a set $T(p)$ for which $p(A) < c(A)$ implies $T(p) \cap A = \emptyset$.
- *Question:* characterize "reasonable" c.
- *Definition.* A sequence of sets $\{X_i\}$ is called $\cup$-*independent* if for all i,

$$X_i \not\subseteq \bigcup_{j \neq i} X_j.$$

- *Theorem.* If c is reasonable and $\{X_i\}$ is $\cup$-independent, then $\sum c(X_i) \leq 1$.
- *Theorem.* If $\sum c(X_i) \leq 1 - \varepsilon$ for all $\cup$-independent definable families $\{X_i\}$, then c is reasonable.

13. Acknowledgments

This work was supported:

- by NASA grant NCC5-209,
- by USAF grant F49620-00-1-0365,
- by NSF grants EAR-0112968, EAR-0225670, and EIA-0321328,
- by Army Research Laboratories grant DATM-05-02-C-0046, and
- by the NIH grant 3T34GM008048-20S1.