Set-Valued Extensions of Fuzzy Logic: Classification Theorems

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Title Page

◀◀   ▶▶

◀   ▶

Page 1 of 14

Go Back

Full Screen

Close

1. Outline

- *Fact:* experts are often not 100% confident.

- *Traditional fuzzy logic:* use numbers from $[0, 1]$.

- *Problem:* an expert often cannot describe degree by a *single* number.

- *Solution:* use a set of numbers.

- *Our result:* the class of such sets coincides:

    - with all 1-point sets (i.e., with the traditional fuzzy logic), or

    - with all subintervals of $[0, 1]$, or

    - with all (closed) subsets of $[0, 1]$.

- *Conclusion:* if we want to go beyond standard fuzzy logic and still avoid sets of arbitrary complexity, we have to use *intervals*.

Title Page

◀◀   ▶▶

◀   ▶

Page 2 of 14

Go Back

Full Screen

Close

2. Fuzzy Logic: Brief Reminder

- *Classical (2-valued) logic:* every statement is either true or false.

- *Problem:* not adequate for expert knowledge, because experts are not fully confident about their statements.

- *Traditional fuzzy logic:* a person's degree of confidence is described by a number from the interval $[0, 1]$:

  - absolute confidence in a statement corresponds to 1,
  - absolute confidence in its negation corresponds to 0.

- *Operations:*

  - *we know:* the degree of confidence a in a statement A and the degree of confidence b in a statement B,
  - *we estimate* the degree of confidence in statements $A \wedge B$ and $A \vee B$ as

    $$a \wedge b \stackrel{\text{def}}{=} \min(a, b) \text{ and } a \vee b \stackrel{\text{def}}{=} \max(a, b).$$

Fuzzy Logic: Brief . . .

Mappings Which . . .

From Single-Valued . . .

Reasonable Properties . . .

It Is Sufficient to . . .

Main Result

Discussion and . . .

Auxiliary Classification . . .

Discussion

Proof: Main Idea

Proof: Main Idea (cont-d)

Acknowledgments

Title Page

◀◀   ▶▶

◀   ▶

Page 3 of 14

Go Back

Full Screen

Close

3. Mappings Which Preserve Standard Fuzzy Logic Operations

- *Important:* there is no absolute scale of degrees.

- *Question:* what possible rescalings $\varphi : [0,1] \to [0,1]$ preserve operations $\wedge$ and $\vee$, in the sense that

  $$\varphi(a) \wedge \varphi(b) = \varphi(a \wedge b) \text{ and } \varphi(a) \vee \varphi(b) = \varphi(a \vee b).$$

- *Known result:* if a bijection (1-1 onto mapping) is monotonic, then it preserves both $\wedge$ and $\vee$.

- *Known result:* vice versa, if a bijection φ preserves the operations $\wedge$ and $\vee$, then it is monotonic.

- *Terminology:* a strictly monotonic continuous function from $[0,1]$ to $[0,1]$ for which $\varphi(0) = 0$ and $\varphi(1) = 1$ is thus an *automorphism* of the structure $([0,1], \wedge, \vee)$.

- The set of all automorphisms is called the *automorphism group* of the structure $([0,1], \wedge, \vee)$.

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Proof: Main Idea (cont-d)

Acknowledgments

4. From Single-Valued Fuzzy Logic to Interval-Valued and Set-Valued Ones

- *Need for sets: reminder.*

  – An expert often cannot describe his or her degree by a single number.

  – It is therefore reasonable to describe this degree by, e.g., a *set* of possible values (e.g., an interval).

- *Operations on sets: motivation:*

  – a set A means that all values $a \in A$ are possible,

  – B means that all the values $b \in B$ are possible;

  – so, the set $A \wedge B$ of possible values of $a \wedge b$ is formed by all the values $a \wedge b$ where $a \in A$ and $b \in B$:

$$A \wedge B \stackrel{\text{def}}{=} \{a \wedge b : a \in A, b \in B\}.$$

- Similarly, $A \vee B \stackrel{\text{def}}{=} \{a \vee b : a \in A, b \in B\}$.

5. Reasonable Properties of Set Extensions

- *Problem:* we want to allow sets from a given class $\mathcal{S}$.

- *We want an extension of the traditional fuzzy logic:* $\mathcal{S}$ must contain all one-element sets.

- *We want invariance:* if $S \in \mathcal{S}$, and $\varphi(x)$ is an automorphism, then the image $\varphi(S) = \{\varphi(s) : s \in S\}$ should also be possible, i.e., $\varphi(S) \in \mathcal{S}$.

- *We want closure* under naturally defined $\wedge$ and $\vee$.

- *Situation:* $S \in \mathcal{S}$, values $s_1, s_2, \ldots, s_k, \ldots$ are all possible (i.e., $s_k \in S$), and $s_k \to s$.

- *Analysis:* no matter how accurately we compute s, we will always find s_k that is indistinguishable from s.

- *Conclusion:* it is natural to assume that this limit value s is also possible, i.e., that every set $S \in \mathcal{S}$ be closed.

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Reasonable Properties . . .

It Is Sufficient to . . .

Main Result

Discussion and . . .

Auxiliary Classification . . .

Discussion

Proof: Main Idea

Proof: Main Idea (cont-d)

Acknowledgments

Title Page

◀◀   ▶▶

◀   ▶

Page 6 of 14

Go Back

Full Screen

Close

6. It Is Sufficient to Consider Closed Classes of Sets

- *Known:* on the class of all bounded closed sets, there is a natural metric – Hausdorff distance $d_H(S, S')$.

- *Definition:* the smallest $\varepsilon > 0$ for which S is contained in the ε-neighborhood of S' and S' is contained in the ε-neighborhood of S.

- *Interpretation:* if $d_H(S, S') \leq \varepsilon$, and we only know the values $s \in S$ and $s' \in S'$ with accuracy ε, then we cannot distinguish between the sets S and S'.

- *Situation:* $S_1, S_2, \ldots, S_k, \ldots$ are all possible ($S_i \in \mathcal{S}$), and $d_H(S_k, S) \to 0$.

- *Analysis:* no matter how accurately we compute the values, we will always find a set S_k that is indistinguishable from the set S (and possible).

- *Conclusion:* the limit set S is also possible, i.e., $\mathcal{S}$ is *closed* under the Hausdorff metric.

7. Main Result

Definition 1. *A class $\mathcal{S}$ of closed non-empty subsets of the interval $[0,1]$ is called a* set-valued extension of fuzzy logic *if it satisfies the following conditions:*

(i) the class $\mathcal{S}$ contains all 1-element sets $\{s\}$, $s \in [0,1]$;

(ii) the class $\mathcal{S}$ is closed under "and" and "or" operations;

(iii) the class $\mathcal{S}$ is closed under arbitrary automorphisms;

(iv) the class $\mathcal{S}$ is closed under Hausdorff metric.

Theorem 1. *Every set-valued extension of fuzzy logic coincides with one of the following three classes:*

- *the class P of all one-point sets $\{s\}$;*

- *the class I of all subintervals $[\underline{s}, \overline{s}] \subseteq [0,1]$ of the interval $[0,1]$;*

- *the class C of all closed subsets S of the interval $[0,1]$.*

8. Discussion and Auxiliary Results

- *Main result in plain English:* if we do not want arbitrarily complex sets, we must restrict ourselves to intervals.

- *We required:* that all single-valued fuzzy sets are possible.

- *Problem:* as we mentioned, single values are not realistic.

- *Question:* what if we do not make this requirement?

- *First case:* the class $\mathcal{S}$ contains a set S which contains neither 0 not 1.

- *Result:* same as before.

- *Remaining case:* every $S \in \mathcal{S}$ contains 0 or 1.

- *Result:* new classification theorem.

Title Page

◀◀   ▶▶

◀   ▶

Page 9 of 14

Go Back

Full Screen

Close

9. Auxiliary Classification Theorem

If every $S \in \mathcal{S}$ contains 0 or 1, then the class $\mathcal{S}$ is a union of one or more of the following classes:

- the class consisting of a single set $\{0\}$;

- the class consisting of a single set $\{1\}$;

- the class consisting of a single interval $[0, 1]$;

- the class I_0 of all subintervals of type $[0, \overline{s}]$;

- the class I_1 of all subintervals of type $[\underline{s}, 1]$;

- the class I_{01} of all sets $S \subseteq [0, 1]$ of the type $[0, \underline{s}] \cup [\overline{s}, 1]$;

- the class C_0 of all closed subsets $S \subseteq [0, 1]$ s.t. $0 \in S$;

- the class C_1 of all closed subsets $S \subseteq [0, 1]$ s.t. $1 \in S$;

- the class C_{01} of all closed subsets $S \subseteq [0, 1]$ which contain both 0 and 1.

Title Page

◀◀ ▶▶

◀ ▶

Page 10 of 14

Go Back

Full Screen

Close

Fuzzy Logic: Brief...

Mappings Which...

From Single-Valued...

Reasonable Properties...

It Is Sufficient to...

Main Result

Discussion and...

Auxiliary Classification...

Discussion

Proof: Main Idea

Proof: Main Idea (cont-d)

Acknowledgments

10. Discussion

- *Particular case:* 3-valued logic: true = 1, false = 0, and unknown = $[0, 1]$.

- *Particular case:* classical logic $\mathcal{S} = \{\{0\}, \{1\}\}$.

- *All other cases:* we have either intervals or arbitrarily complex closed set.

- *Conclusion:* if we do not want arbitrarily complex sets, we must restrict ourselves to intervals.

- *Natural generalization:* from sets to fuzzy numbers (type-2 fuzzy sets). Result:
    - if $\mathcal{S}$ contains *at least one* strictly monotonic fuzzy number and
    - $\mathcal{S}$ is invariant under automorphisms and Hausdorff closed,
    - then $\mathcal{S}$ contains *all* fuzzy numbers.

Title Page

◀◀   ▶▶

◀   ▶

Page 11 of 14

Go Back

Full Screen

Close

Title Page

◀◀   ▶▶

◀   ▶

Page 12 of 14

Go Back

Full Screen

Close

11. Proof: Main Idea

- It is enough to consider case when $\mathcal{S}$ contains at least one non-1-point set.

- If $\mathcal{S}$ contains a non-degenerate interval $[\underline{s}, \overline{s}]$, then automorphisms can move it close to $[0, 1]$.

- Due to Hausdorff-closeness, $[0, 1] \in \mathcal{S}$.

- Hence $[a, b] = ([0, 1] \vee \{a\}) \wedge \{b\} \in \mathcal{S}$ for all $a < b$.

- Let $S \in \mathcal{S}$ be not an interval, then $s_0 \notin S$ for some $s_0 \in [\inf S, \sup S]$.

- Since S is closed, the whole neighborhood $(s_0 - \varepsilon, s_0 + \varepsilon)$ is outside S.

- An automorphism can move $s_0 - \varepsilon$ close to 0, and $s_0 + \varepsilon$ close to 1.

- Due to Hausdorff-closeness, $\{0, 1\} \in \mathcal{S}$.

12.   Proof: Main Idea (cont-d)

- We have just proved: $\{0, 1\} \in \mathcal{S}$.

- Hence, $\{0, 1\} \wedge \{p_n\} = \{0, p_n\} \in \mathcal{S}$.

- Here,

$$\{0, p_{n-k}, p_{(n-k)+1}, \ldots, p_n\} \vee \{0, p_{n-k-1}\} =$$

$$\{0, p_{n-k-1}, p_{n-k}, p_{(n-k)+1}, \ldots, p_n\}.$$

- So, by induction over k, we have $\{0, p_1, p_2, \ldots, p_n\} \in \mathcal{S}$.

- Hence, $\{p_1, p_2, \ldots, p_n\} = \{p_1\} \vee \{0, p_1, p_2, \ldots, p_n\} \in \mathcal{S}$.

- *Conclusion:* every finite set is in $\mathcal{S}$.

- *Known fact:* every closed set is a limit of finite sets.

- *Conclusion:* every closed set is in $\mathcal{S}$.

Title Page

◀◀   ▶▶

◀   ▶

Page 13 of 14

Go Back

Full Screen

Close

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