Everything Is a Matter of Degree: A New Theoretical Justification of Zadeh's Principle

Hung T. Nguyen
Department of Mathematical Sciences
New Mexico State University

Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968
Email: vladik@utep.edu

Everything Is a Matter . . .
Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

1. Everything Is a Matter of Degree: One of the Main Ideas Behind Fuzzy Logic

- One of the main ideas behind Zadeh's fuzzy logic and its applications is that everything is a matter of degree.
- We are often accustomed to think that every statement about a physical world is true or false:
  - that an object is either a particle or a wave,
  - that a person is either young or not,
  - that a person is either well or ill.
- However, in reality, we sometimes encounter intermediate situations.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

2. Formulation of the Problem

- That everything is a matter of degree is a convincing empirical fact.
- A natural question is: why?
- How can we explain this fact?
- This is what we will try to do in this talk: come up with a theoretical explanation of this empirical fact.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

3. There Should be an Objective Theoretical Explanation for Fuzziness

- Most traditional examples of fuzziness come from the analysis of commonsense reasoning.
- When we reason, we use words from natural language like "young", "well".
- In many practical situations, these words do not have a precise true-or-false meaning, they are fuzzy.
- Impression: fuzziness is subjective, it is how our brains work.
- However, we are the result of billions of years of successful adjusting-to-the-environment evolution.
- Everything about us humans is not accidental.
- In particular, the fuzziness in our reasoning must have an objective explanation – in fuzziness of the real world.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

4. First Example of Objective "Fuzziness" – Fractals

- Since the ancient times, we know:
  - 0-dimensional objects (points),
  - 1-dimensional objects (lines),
  - 2-dimensional objects (surfaces),
  - 3-dimensional objects (bodies), etc.
- In all these examples, dim is an integer: 0, 1, 2, 3, etc.
- In the 19th century, mathematicians discovered sets of fractional dimension (fractals).
- In the 1970s, B. Mandlebrot noticed that many real-life objects are fractals, e.g.:
  - shoreline of England
  - shape of the clouds and mountains
  - noises in electric circuits.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

5. Second Example of Objective "Fuzziness" – Quantum Physics

- In general, states are described by *continuous* variables.
- However, the set of stable states is usually *discrete*.
- *Example:* computers use memory cells with 2 stable states representing 0 and 1.
- *In quantum physics:* we can have superpositions $c_0 \cdot \langle 0| + c_1 \cdot \langle 1|$ for complex c_i.
- Resulting quantum computations are much faster:
  - we can search in an unsorted list of n elements in time $\sqrt{n}$;
  - we can factor large integers fast – and thus, crack the existing codes.
- What we originally thought of as an integer-valued variable turned out to be real-valued.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

6. Third Example of Objective "Fuzziness" – Fractional Charges of Quarks

- Matter is seemingly continuous.
- It turned out that matter is discrete: it consists of molecules, atoms, and elementary particles.
- One experimental fact: all electric charges are proportional to a single charge.
- Thus, protons, etc., cannot be further decomposed.
- Gell-Mann discovered that we can design p, n, mesons, etc. in terms of a few *quarks*.
- Interesting aspect: quarks have fractional electric charge.
- Original idea: quarks are theoretical concepts.
- Experiments revealed 3 *partons* within p – actual quarks.
- So, what we originally thought of as an integer-valued variable turned out to be real-valued.

Formulation of the ...
There Should be an ...
First Example of ...
Second Example of ...
Third Example of ...
Our Explanation of ...
First Explanation: ...
Second Explanation: ...
Symmetry: Another ...
Case Study: Territory ...
Acknowledgments

7. Our Explanation of Why Physical Quantities Originally Thought to Be Integer-Valued Turned out to Be Real-Valued: Main Idea

- *In philosophical terms:* what we are doing is "cognizing" the world.
- *Clarification:* understanding how it works and trying to predict consequences of different actions.
- *Objective:* select the most beneficial action.
- If a phenomenon is not cognizable, there is nothing we can do about it.
- *Our explanation:* in cognizable phenomena, it is reasonable to expect continuous-valued variables.
- *In other words:* properties originally thought to be discrete are actually matters of degree.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

8. First Explanation: Goedel's Theorem vs. Tarski's Algorithm

- *Goedel's theorem:* 1st example of non-cognizability.
- *Formulations:*
  - variables x, y, z, etc. run over integers;
  - terms t are formed from x, …, and const. by $+$, $\cdot$;
  - elementary formulas: $t = t'$, $t < t'$, $t \le t'$, etc.
  - formulas: from elem. formulas by $\vee$, $\&$, $\neg$, $\exists$, $\forall$.
- *Example:* $\forall x\, \forall y (x < y \rightarrow \exists z (y = x + z))$.
- *Goedel's theorem:* no algorithm can tell whether a given formula is true or not.
- *Tarski's theorem:* if we consider variables over real numbers, then such an algorithm is possible.
- *Conclusion:* in cognizable situations, we must have continuous-valued variables.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

9. Second Explanation: Efficient Algorithms for Linear Algebra vs. NP-Hardness of Integer Programming

- *Practical situation:* find the values $x_1, \ldots, x_n$ from the results $y_1, \ldots, y_m$ of indirect measurements:

$$f_1(x_1, \ldots, x_n) = y_1; \quad f_m(x_1, \ldots, x_n) = y_m.$$

- *Frequent case:* we know approximate $\widetilde{x}_i$ values of x_i.
- *How this helps:* we can linearize the system:

$$a_{i1} \cdot \Delta x_1 + \ldots + a_{in} \cdot \Delta x_n = \Delta y_i, \quad 1 \le i \le m.$$

- *Case of continuous variables:* efficient algorithms solve systems of linear equations.
- *Case of discrete variables:* problem becomes NP-hard.
- *Meaning (informal):* every algorithm requires un-realistic time in some cases (unless P=NP).

Formulation of the ...
There Should be an ...
First Example of ...
Second Example of ...
Third Example of ...
Our Explanation of ...
First Explanation: ...
Second Explanation: ...
Symmetry: Another ...
Case Study: Territory ...
Acknowledgments

10. Symmetry: Another Fundamental Reason for Continuity ("Fuzziness")

- *Case study: benzene* C_6H_6.
  - circular arrangement came to Kekule in a dream;
  - *analysis:* C has valency 4, 1 is connected to H;
  - *hence:* 3 connections for two C neighbors;
  - *result:* 2- and 1-connections interchange;
  - *in reality:* all connections are equivalent;
  - *explanation:* quantum "valency" 3/2.
- *Case study: fuzzy logic.*
  - *complete uncertainty* means that we have exactly the same degree of belief in A and in $\neg A$;
  - *in traditional (2-valued) logic:* there is no truth value invariant under negation $A \rightarrow \neg A$;
  - *in fuzzy logic:* 0.5 is such a value.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

11. Case Study: Territory Division

- *Problem:* divide a disputed territory T between n parties: $T = T_1 \cup \ldots \cup T_n$.
- *Traditional description:* maximize Nash's criterion $U_1 \cdot \ldots \cdot U_n$, where i-th utility is $U_i = \int_{T_i} u_i(x)\,dx$.
- *Solution:* for some weights c_i, a point x goes to the party with the largest utility $c_i \cdot u_i(x)$.
- *Natural question:* why not joint control?
- *Formalization:* select $d_i(x)$ s.t. $d_1(x) + \ldots + d_n(x) = 1$, then $U_i = \int d_i(x) \cdot u_i(x)\,dx$.
- *First result:* this problem always has a crisp division.
- *Additional requirement:* the solution should preserve the problem's symmetry.
- *Second result:* in some cases – e.g., when $u_1(x) = \ldots = u_n(x) = \text{const}$ – only fuzzy divisions are optimal.

Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments

12. Acknowledgments

This work was supported in part:

- by NSF grants HRD-0734825, EAR-0225670, and EIA-0080940,
- by Texas Department of Transportation contract No. 0-5453,
- by the Japan Advanced Institute of Science and Technology (JAIST) International Joint Research Grant 2006-08, and
- by the Max Planck Institut für Mathematik.

The authors are thankful to the anonymous referees for valuable suggestions.

Everything Is a Matter . . .
Formulation of the . . .
There Should be an . . .
First Example of . . .
Second Example of . . .
Third Example of . . .
Our Explanation of . . .
First Explanation: . . .
Second Explanation: . . .
Symmetry: Another . . .
Case Study: Territory . . .
Acknowledgments