Empirical Formulas for Economic Fluctuations: Towards a New Justification

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1. It Is Important to Take into Account Economic Fluctuations

- *Fact:* stock prices (and other related economic indices) fluctuate in an unpredictable ("random") way.
- *Usually:* these fluctuations are small.
- *Sometimes:* the fluctuations become large.
- *Crises:* large negative fluctuations bring havoc to the economy and finance, lead to crisis situations.
- *Consequence:* it is therefore important to correctly take such fluctuations into account.
- *Problem:* in particular, it is extremely important to correctly predict the probability of large fluctuations.

2. Gaussian Random Walk Model

- *Pioneering work:* L. Bachelier's PhD (1900).
- *Formula:* fluctuations of different sizes x are normally distributed, with pdf $\rho(x) = \dfrac{1}{\sqrt{2\pi}\cdot\sigma}\cdot\exp\left(-\dfrac{x^2}{2\sigma^2}\right)$.
- *Fact:* The random walk model indeed describes small fluctuations reasonably well.
- *Problem:* this model drastically underestimates the probabilities of *large* fluctuations:
  - in the normal distribution, fluctuations larger than 6σ have a negligible probability $\approx 10^{-8}$, while
  - in real economic systems, even larger fluctuations occur every decade (and even more frequently).
- *Consequences:* we thus underestimate risk – and become unprepared when large fluctuations occur.

3. Mandelbrot's Fractal Model

- *Pioneer:* Benoit Mandelbrot (1960s), the father of fractals.
- *Empirical results:* medium-scale fluctuations follow the power-law distribution $\rho(x) = A \cdot x^{-\alpha}$ for $\alpha \approx 2.7$.
- *Problem:* this model drastically *over*estimates the probability of large-scale fluctuations.
- *Indeed:* $P(x > x_0) \sim \dfrac{1}{x_0^{1.7}}$, hence $\dfrac{P(x > x_0)}{P(x > X_0)} \approx \left(\dfrac{X_0}{x_0}\right)^{1.7}$.
- Daily fluctuations of $\approx x_0 = 1\%$ are normal, with probability $P \approx 1$.
- Thus, the probability of a crisis is
$$P(x > X_0) \approx \frac{1}{30^{1.7}} \approx \frac{1}{300}.$$
- *Consequence:* we should have crises every year.

4. Empirical Analysis of Economic Fluctuations (Econophysics)

- *Empirical result:* large fluctuations are distributed with $\rho(x) = A \cdot x^{-4}$, so $P(x > x_0) \sim \dfrac{1}{x_0^3}$ ("cubic law").
- *Fact:* in the practical financial engineering applications, this cubic law is rarely used.
- *Main reason:* the cubic law lacks a clear theoretical justification.
- *Clarification:* existing explanations depend on complex math assumptions – and are not clear to economists.
- *Consequence:* prevailing economic models mis-estimate probabilities of large fluctuations.
- *Our objective:* provide simpler – and hopefully more convincing – explanations for the cubic law.

5. A Practice-Oriented Temporal Reformulation of the Probabilities

- *Ideally:* we should be able to predict when the fluctuations will reach a given size x_0.
- *In reality:* economic fluctuations are random (unpredictable).
- *Conclusion:* we can only predict the *average* time $t(x_0)$ before such a fluctuation occurs.
- *Relation to $\rho(x_0)$:* during the time period t, we have $N \stackrel{\text{def}}{=} \dfrac{t}{\Delta t}$ time quanta, hence $\dfrac{t}{\Delta t} \cdot (\rho(x_0) \cdot h)$ fluctuations.
- *Conclusion:* $t(x_0)$ is when we have one fluctuation: $t(x_0) \approx \dfrac{\Delta t}{\rho(x_0) \cdot h}$.
- *Vice versa:* once we find $t(x)$, we get $\rho(x) \approx \dfrac{\text{const}}{t(x)}$.

6. Scale Invariance: A Natural Requirement

- *Fact:* the numerical value of the fluctuation size x depends on the choice of a measuring unit.
- *Example:* when European countries switched to Euros, all the stock prices were re-scaled $x \to x' = \lambda \cdot x$.
- *Reasonable requirement:* $t(x)$ should not depend on the choice of the unit.
- *Clarification:* the fluctuation of 0.1 Euros will happen faster than a fluctuation of 1 Euro.
- *Clarified requirement:* we have to also re-scale time.
- *Resulting requirement:* for every $\lambda > 0$, there exists a value $r(\lambda)$ for which, for all x and for all λ, we have

$$t(\lambda \cdot x) = r(\lambda) \cdot t(x).$$

7. Scale-Invariance Implies Power Law

- *Scale invariance* (reminder): $t(\lambda\cdot x)=r(\lambda)\cdot t(x)$.
- *Step 1:* differentiate w.r.t. λ and take $\lambda=1$:

$$x\cdot\frac{dt}{dx}=\alpha\cdot t,\ \text{where } \alpha\stackrel{\text{def}}{=}r'(1).$$

- *Step 2:* separate variables:

$$\frac{dt}{t}=\alpha\cdot\frac{dx}{x}.$$

- *Step 3:* integrate:

$$\ln(t)=\alpha\cdot\ln(x)+c.$$

- *Step 4:* exponentiate:

$$t=C\cdot x^{\alpha},\ \text{hence } \rho(x)\sim\frac{\text{const}}{t(x)}\sim x^{-\alpha}.$$

8. Individual Stock: Idealized Case

- *Remaining question:* what is α?
- *Up to now:* we considered fluctuations which occur within a single time quantum Δt.
- *Possibility:* we can consider different time quanta: e.g., a time quantum $\Delta t' = k \cdot \Delta t$ for some integer k.
- *Reminder:* price fluctuations are reasonably accurately described by a random walk.
- *Scaling for a random walk:* fluctuation over $\Delta t' = k \cdot \Delta t$ is $\sqrt{k}$ times larger than for Δt.
- When $t \to k \cdot t$, we get $x \to \sqrt{k} \cdot x$.
- In terms of λ and $r(\lambda)$, this means that when $\lambda = \sqrt{k}$, we have $r(\lambda) = k$, i.e., $r(\lambda) = \lambda^2$.
- For $t(x) = C \cdot x^\alpha$, the requirement $t(\lambda \cdot x) = r(\lambda) \cdot t(x)$ leads to $\alpha = 2$ and $\rho(x) \sim x^{-2}$, i.e., $P(x > x_0) \sim x_0^{-1}$.

9. From the Idealized Case of an Individual Stock to Stock Market: Two Approaches

- We considered the case of an individual stock which is not interacting with other stock prices.
- In reality, stocks are inter-related: a change in one stock price causes a change in prices of other stocks.
- How can we take this dependence into account?
- In this talk, we describe two approaches for taking this dependence into account:
  - a probabilistic approach, and
  - a fuzzy approach.
- We will show that both approaches lead to the same distribution.
- This makes us even more confident that this is indeed a correct distribution.

10. Probabilistic Approach

- Stocks are usually classified based on 4 characteristics:
  - the size of the company (large cap vs. mid cap vs. small cap stocks);
  - the size of the dividend (income stocks vs. non-income ones);
  - cyclicity (cyclic stocks vs. defensive stocks);
  - stability (less risky value stocks vs. more aggressive and more risky growth stocks).
- Crudely speaking, this means that we have 4 different extreme types of stocks.
- Every stock is, in some reasonable sense, equivalent to a combination of these different 4 types.
- *Fact:* different types of stocks behave independently from each other.

11. Probabilistic Approach (cont-d)

- *Comment:* independence is used to explain the balanced investment portfolio.
- Usually, individual stock price changes largely compensate each other.
- For a stock market index to really change, the majority of stocks must experience the drastic change.
- "The majority" means that at least three types of stock out of four must experience a drastic fluctuation.
- For individual stock, the probability of a fluctuation is $\sim x_0^{-1}$.
- Due to independence, the probability of a stock market fluctuation is

$$P(x > x_0) \sim x_0^{-1} \cdot x_0^{-1} \cdot x_0^{-1} = x_0^{-3}.$$

12. Fuzzy Approach

- For *large* (but reasonable size) fluctuations, we can use the individual stock description, w/pdf $\rho(x) \sim x^{-2}$.
- Correspondingly, a reasonable membership function $\mu(x)$ can be obtained by normalizing this expression:

$$\mu_{\text{large}}(x) = \frac{\rho(x)}{\max\limits_{y} \rho(y)} \sim x^{-2}.$$

- We are interested in *very large* (crisis) fluctuations.
- In fuzzy logic, the most widely used way to describe "very" is to take the square:

$$\mu_{\text{very\_large}}(x) = \mu^2_{\text{large}}(x) \sim (x^{-2})^2 = x^{-4}.$$

- Thus, the corresponding probability density function is also proportional to x^{-4}.
- So, we have indeed justified the cubic law.

13. Discussion

- We have justified the same empirical distribution by using two different approaches:
  - the probabilistic approach and
  - the fuzzy approach.
- By their very origins,
  - the probabilistic approach is usually based on the (more) mathematical analysis, while
  - the fuzzy approach is more oriented towards natural language and commonsense reasoning.
- In perfect accordance with this difference, the derivation is much clearer when we use fuzzy logic.

14. Conclusion

- *Objective:* to enhance the use of accurate empirical descriptions of economic fluctuations.
- *Means:* it is necessary to provide a theoretical justification for these empirical descriptions.
- *What we did:* we provided two such justifications, based on
  - probabilistic approach and
  - fuzzy approach.
- *Fact:* both justifications lead to the same distribution.
- *Conclusion:* this fact further increases our confidence in this empirical distribution.

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