# How to Relate Fuzzy and OWA Estimates

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# 1. Single-Quantity Data Fusion: A Problem

- In many practical situations, we have several estimates $x_1, \ldots, x_n$ of the same quantity $x$:

$$x_1 \approx x, \quad x_2 \approx x, \quad \ldots, \quad x_n \approx x.$$

- It is desirable to combine (fuse) these estimates into a single estimate for $x$.
- From the fuzzy viewpoint, a natural way to combine these estimates is as follows:
  - to describe, for each $x$ and for each $i$, the degree $\mu_\approx(x_i - x)$ to which $x$ is close to $x_i$;
  - to use a t-norm ("and"-operation) $t_\&(a, b)$ to combine these degrees into a single degree

$$d(x) = t_\&(\mu_\approx(x_1 - x), \ldots, \mu_\approx(x_n - x));$$

  - and find the estimate $x$ for which the degree $d(x)$ – that $x$ is close to all $x_i$ – is the largest.

# 2. Limitation of Fuzzy and Emergence of OWA

- *Reminder:* find $x$ that maximizes
$$d(x) = t_\&(\mu_\approx(x_1 - x), \ldots, \mu_\approx(x_n - x)).$$
- *Main problem:* the corresponding procedure is computationally complex, esp. for generic $\mu_\approx(x)$ and $t_\&(a, b)$.
- *Solution:* OWA (Ordered Weighted Average) approach:
  - sort the values $x_1, \ldots, x_n$ into an increasing sequence
  $$x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)};$$
  - select the weights $w_1, \ldots, w_n \ge 0$ for which
  $$\sum_{i=1}^n w_i = 1;$$
  - use the weighted average $x = \sum\limits_{i=1}^n w_i \cdot x_{(i)}$ as the desired fused estimate.

## 3. Formulation of the Problem

- To get a better fusion:
  - we must appropriately select the membership function $\mu_{\approx}(x)$ and the t-norm (in the fuzzy case), and
  - we must appropriately select the weights $w_i$ (in the OWA case).
- Both approaches – when applied properly – lead to reasonable data fusion.
- It is therefore desirable to be able to relate the corresponding selections:
  - once we have found the appropriate $\mu_{\approx}(x)$ and t-norm, we should be able to deduce the weights;
  - once we have found the appropriate weights, we should be able to deduce $\mu_{\approx}(x)$ and t-norm.

## 4. Reducing to the Case of Archimedean t-Norms

- *Archimedean* t-norms have the form

$$t_\&(a,b) = f^{-1}(f(a)\cdot f(b)).$$

- *It is known* that a general t-norm can be obtained:
  - by setting Archimedean t-norms on several (maybe infinitely many) subintervals of the interval $[0,1]$,
  - by using $\min(a,b)$ as the value of $t_\&(a,b)$ for the cases when $a$ and $b$ are not in the same interval.
- *Conclusion:* for every t-norm and for every $\varepsilon > 0$, there exists an $\varepsilon$-close Archimedean t-norm.
- *Idea of the proof:* replace min with a close Archimedean t-norm, e.g., with $(a^{-p}+b^{-p})^{-1/p}$ for a large $p$.
- So, from the practical viewpoint, we can always safely assume that the t-norm is Archimedean.

## 5. Fuzzy Fusion for Archimedean t-Norms

- *Reminder:* we maximize

$$d(x) = t_\&(\mu_\approx(x_1 - x), \ldots, \mu_\approx(x_n - x)).$$

- *Archimedean t-norm:* $t_\&(a, b) = f^{-1}(f(a) \cdot f(b))$, so

$$d(x) = f^{-1}(f(\mu_\approx(x_1 - x)) \cdot \ldots \cdot f(\mu_\approx(x_n - x))).$$

- *Fact:* $d(x) \to \max \Leftrightarrow D(x) \stackrel{\text{def}}{=} f(d(x)) \to \max$, where

$$D(x) = f(\mu_\approx(x_1 - x)) \cdot \ldots \cdot f(\mu_\approx(x_n - x)).$$

- *Alternative description:*

$$D(x) = \prod_{i=1}^{n} \rho(x_i - x),$$

where $\rho(x) \stackrel{\text{def}}{=} f(\mu_\approx(x))$.

# 6. Resulting Reformulation of the Problem

- We have two ways to fuse estimates $x_1, \ldots, x_n$ into a single estimate $x$:
  - find $x$ for which the value $\prod_{i=1}^{n} \rho(x_i - x)$ is the largest possible (*fuzzy approach*), and
  - find $x$ as $\sum_{i=1}^{n} w_i \cdot x_{(i)}$ (*OWA approach*).
- The problem is:
  - given $\rho(x)$, find $w_i$ for which the OWA estimate is close to the original fuzzy estimate; and
  - given $w_i$, find $\rho(x)$ for which the fuzzy estimate is close to the original OWA estimate.

# 7. A Similar Problem Is Already Solved In Robust Statistics

- *Robust statistics:* making estimates under partial information about the probability distribution $f(x)$.
- *Typical techniques:* use statistical techniques corresponding to some pdf $f_0(x)$.
- *M-methods:* Max Likelihood

$$\prod_{i=1}^{n} f_0(x_i - a) \to \max_a .$$

- *L-estimates:* $a_L = \frac{1}{n} \cdot \sum_{i=1}^{n} m\left(\frac{i}{n}\right) \cdot x_{(i)}$ for some $m(p)$.
- *Observation:* these are exactly our formulas for fuzzy and OWA estimates, with

$$\rho(x) = f_0(x) \text{ and } w_i = \frac{1}{n} \cdot m\left(\frac{i}{n}\right).$$

# 8. Relation between M-methods and L-Estimates

- *Reminder:* we have estimates:
  - $a_m$ s.t. $\sum\limits_{i=1}^{n} f_0(x_i - a_M) \to \max\limits_a$, and
  - $a_L = \dfrac{1}{n} \cdot \sum\limits_{i=1}^{n} m\left(\dfrac{i}{n}\right) \cdot x_{(i)}$.
- *Fact:* in robust statistics, it is known how, given $f_0(x)$, to find $m(p)$ for which $a_M$ and $a_L$ are asympt. close:
  - we compute the cumulative distribution function $F_0(x)$ as $F_0(x) = \int_{-\infty}^{x} f_0(t)\,dt$;
  - we find the auxiliary function $M(p) = z(F_0^{-1}(p))$, where $z(x) \stackrel{\text{def}}{=} -(\ln(f_0(x))''$;
  - we normalize $m(p) = \dfrac{M(p)}{\int_0^1 M(q)\,dq}$.
- *Our idea:* use this relation to compare fuzzy and OWA estimates.

# 9. M-methods vs. L-Estimates: Example

- *Reminder:*
  - we compute cdf $F_0(x) = \int_{-\infty}^{x} f_0(t)\,dt$;
  - we find $M(p) = z(F_0^{-1}(p))$, where $z(x) \stackrel{\text{def}}{=} -(\ln(f_0(x))''$;
  - we compute $m(p) = \dfrac{M(p)}{\int_0^1 M(q)\,dq}$.
- The Gaussian function $f_0(x) = \exp\left(-\frac{1}{2}\cdot x^2\right)$ is proportional to the pdf of the normal distribution.
- Hence, $F_0(x) = \int_{-\infty}^{x} f_0(t)\,dt$ is proportional to the cdf of a normal distribution.
- Here, $\ln(f_0(x)) = -\frac{1}{2}\cdot x^2$, hence
$$z(x) = -\ln(f_0(x))'' = 1.$$
- So, $M(p) = z(F_0^{-1}(p)) = 1$; the integral of $M(p) = 1$ over the interval $[0,1]$ is 1, hence $m(p) = M(p) = 1$.

## 10. Relation Between Fuzzy and OWA Estimates: Our Main Idea

- We have seen that, mathematically,
  - M-estimates correspond to fuzzy estimates, and
  - L-estimates correspond to OWA estimates.
- We can therefore
  - use the solution provided by robust statistics
  - to find the desired correspondence between the utility function and the spectral risk measures.

# 11. Resulting Solution: from Fuzzy to OWA

- We start with the functions that describe a fuzzy estimate.
- Specifically, we have functions $\mu_{\approx}(x)$ and $f(x)$ for which
$$t_{\&}(a,b) = f^{-1}(f(a)\cdot f(b)).$$
- We compute an auxiliary function $f_0(x) = f(\mu_{\approx}(x))$.
- Then, we compute the second auxiliary function
$$F_0(x) = \int_{-\infty}^{x} f_0(t)\,dt.$$
- After that, we find the third auxiliary function
$$M(p) = z(F_0^{-1}(p)), \text{ where } z(x) = -(\ln(f_0(x))''.$$
- Finally, we compute $I \stackrel{\text{def}}{=} \int_0^1 M(q)\,dq$, then
$$m(p) = \frac{M(p)}{I} \text{ and } w_i = \frac{1}{n}\cdot m\left(\frac{i}{n}\right).$$

# 12. From OWA to Fuzzy

- *Situation:* we know the weights $w_i$, and we want to find the membership function and the t-norm.
- First, by extrapolation, we find a function $m(p)$ for which $m\left(\frac{i}{n}\right) = n \cdot w_i$.
- Then, we find the auxiliary function $F_0(x)$ and the auxiliary value $I$ by solving the equation

$$I \cdot m(F_0(x)) = -(\ln(F_0'(x)))''.$$

- After that, we find $f_0(x) = F_0'(x)$.
- For a general Archimedean t-norm $t_\&(a,b)$, we first find the function $f(x)$ for which $t_\&(a,b) = f^{-1}(f(a)\cdot f(b))$.
- Then, from the equality $f_0(x) = f(\mu_\approx(x))$, we conclude that $\mu_\approx(x) = f^{-1}(f_0(x))$.

# 13. Example

- *Case:* Gaussian $\mu_{\approx}(x)$ and $t_{\&}(a,b) = a \cdot b$ ($f(x) = x$).
- *Analysis:* the condition $\prod\limits_{i=1}^{n} \rho(x_i - x) \to \max\limits_{x}$ leads to

$$\Pi \stackrel{\text{def}}{=} \prod_{i=1}^{n} \exp\left(-\frac{1}{2} \cdot (x_i - x)^2\right) \to \max_{x}$$

.

- $\Pi \to \max_x \Leftrightarrow -\ln(\Pi) = \dfrac{1}{2} \cdot \sum\limits_{i=1}^{n} (x_i - x)^2 \to \min\limits_{x}$.
- *Solution:* $x = \dfrac{1}{n} \cdot \sum\limits_{i=1}^{n} x_i$, i.e., $w_i = \dfrac{1}{n}$.
- When we apply the above algorithm to these $\mu_{\approx}(x)$ and $f(x) = x$, we indeed get $m(p) = 1$ and $w_i = \dfrac{1}{n}$.

## 14. Acknowledgment

- This work was also supported in part
  - by the National Science Foundation grants HRD-0734825 and DUE-0926721,
  - by Grant 1 T36 GM078000-01 from the National Institutes of Health, and
  - by the Science and Technology Centre in Ukraine (STCU) Grant 5015, funded by European Union.
- The authors are thankful to Hung T. Nguyen for valuable discussions.