Fundamental Physical Equations Can Be Derived By Applying Fuzzy Methodology to Informal Physical Ideas

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1. Introduction

- Fuzzy methodology has been invented to transform:
  - expert ideas – formulated in terms of words from natural language,
  - into precise rules and formulas, rules and formulas understandable by a computer.
- Fuzzy methodology: main success is intelligent (fuzzy) control.
- We show that the same fuzzy methodology can also lead to the exact fundamental equations of physics.
- This fact provides an additional justification for the fuzzy methodology.

2. Newton's Physics: Informal Description

- A body usually tries to go to the points x where its potential energy $V(x)$ is the smallest.
- For example, a moving rock on the mountain tries to go down.
- The sum of the potential energy $V(x)$ and the kinetic energy K is preserved:

$$K = \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt} \right)^2 .$$

- Thus, when the body minimizes its potential energy, it thus tries to maximize its kinetic energy.
- We will show that when we apply the fuzzy techniques to this informal description, we get Newton's equations

$$m \cdot \frac{d^2 x_i}{dt^2} = - \frac{\partial V}{\partial x_i}.$$

3. First Step: Selecting a Membership Function

- The body tries to get to the areas where the potential energy $V(x)$ is small.
- We need to select the corresponding membership function $\mu(V)$.
- For example, we can poll several (n) experts and if $n(V)$ of them consider V small, take $\mu(V) = \dfrac{n(V)}{n}$.
- In physics, we only know *relative* potential energy – relative to some level.
- If we change that level by V_0, we replace V by $V + V_0$.
- So, values V and $V + V_0$ represent the same value of the potential energy – but for different levels.
- A seemingly natural formalization: $\mu(V) = \mu(V + V_0)$.
- Problem: we get useless $\mu(V) = \text{const}$.

4. Re-Analyzing the Polling Method

- In the poll, the more people we ask, the more accurate is the resulting opinion.
- Thus, to improve the accuracy of the poll, we add m folks to the original n top experts.
- These m extra folks may be too intimidated by the original experts.
- With the new experts mute, we still have the same number $n(V)$ of experts who say "yes".
- As a result, instead of the original value $\mu(V)=\dfrac{n(V)}{n}$, we get $\mu'(V)=\dfrac{n(V)}{n+m}=c\cdot\mu(V)$, where $c=\dfrac{n}{n+m}$.
- These two membership functions $\mu(V)$ and $\mu'(V)=c\cdot\mu(V)$ represent the same expert opinion.

5. Resulting Formalization of the Physical Intuition

- *How* to describe that potential energy is small?
- *Idea:* value V and $V + V_0$ are equivalent – they differ by a starting level for measuring potential energy.
- *Conclusion:* membership functions $\mu(V)$ and $\mu(V+V_0)$ should be equivalent.
- *We know:* membership functions $\mu(V)$ and $\mu'(V)$ are equivalent if $\mu'(V) = c \cdot \mu(V)$.
- *Hence:* for every V_0, there is a value $c(V_0)$ for which

$$\mu(V + V_0) = c(V_0) \cdot \mu(V).$$

- *It is known* that the only monotonic solution to this equation is $\mu(V) = a \cdot \exp(-k \cdot V)$.
- *So* we will use this membership function to describe that the potential energy is small.

6. Resulting Formalization of the Physical Intuition (cont-d)

- *Reminder:* we use $\mu(V) = a \cdot \exp(-k \cdot V)$ to describe that potential energy is small.
- *How* to describe that kinetic energy is large?
- *Idea:* K is large if $-K$ is small.
- *Resulting membership function:*

$$\mu(K) = \exp(-k \cdot (-K)) = \exp(k \cdot K).$$

- *We want* to describe the intuition that
  - the potential energy is small *and*
  - that the kinetic energy is large *and*
  - that the same is true at different moments of time.
- According to fuzzy methodology, we must therefore select an appropriate "and"-operation (t-norm) $f_{\&}(a, b)$.

7. How to Select an Appropriate t-Norm

- In principle, if we have two completely independent systems, we can consider them as a single system.
- Since these systems do not interact with each other, the total energy E is simply equal to $E_1 + E_2$.
- We can estimate the smallness of the total energy in two different ways:
  - we can state that the total energy $E = E_1 + E_2$ is small: certainty $\mu(E_1 + E_2)$, or
  - we can state that both E_1 *and* E_2 are small:

$$f_\&(\mu(E_1), \mu(E_2)).$$

- It is reasonable to require that these two estimates coincide: $\mu(E_1 + E_2) = f_\&(\mu(E_1), \mu(E_2))$.
- This requirement enables us to uniquely determine the corresponding t-norm: $f_\&(a_1, a_2) = a_1 \cdot a_2$.

8. Resulting Model

- *Idea:* at all moments of time $t_1, \ldots, t_N$, the potential energy V is small, and the kinetic energy K is large.
- Small is $\exp(-k \cdot V)$, large is $\exp(k \cdot K)$, "and" is product, thus the degree $\mu(x(t))$ is

$$\mu(x(t)) = \prod_{i=1}^{N} \exp(-k \cdot V(t_i)) \cdot \prod_{i=1}^{N} \exp(k \cdot K(t_i)).$$

- So, $\mu(x(t)) = \exp(-k \cdot S)$, w/ $S \stackrel{\text{def}}{=} \sum\limits_{i=1}^{N} (V(t_i) - K(t_i))$.
- In the limit $t_{i+1} - t_i \to 0$, $S \to \int (V(t) - K(t))\, dt$.
- The most reasonable trajectory is the one for which $\mu(x(t)) \to \max$, i.e., $S = \int L\, dt \to \min$, where

$$L \stackrel{\text{def}}{=} V(t) - K(t) = V(t) - \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2.$$

9. This Model Leads to Newton's Equations

- *Reminder:* $S = \int L\,dt \to \min$, where

$$L \stackrel{\text{def}}{=} V(t) - K(t) = V(t) - \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2.$$

- Most physical laws are now formulated in terms of the *Principle of Least Action* $S = \int L\,dt \to \min$.
- E.g., for the above L, we get Newtonian physics.
- So, *fuzzy indeed implies Newton's equations.*
- *Newton's physics:* only one trajectory, with $S \to \min$.
- With the *fuzzy* approach, we also get the degree $\exp(-k \cdot S)$ w/which other trajectories are reasonable.
- In *quantum* physics, each non-Newtonian trajectory is possible with "amplitude" $\exp(-k \cdot S)$ (for complex k).
- This makes the above derivation even more interesting.

10. Beyond the Simplest Netwon's Equations

- In our analysis, we assume that the expression for the potential energy field $V(x)$ is given.
- In reality, we must also find the equations that describe the corresponding field.
- Simplest case: gravitational field.
- The gravitational pull of the Earth is caused by the Earth as a whole.
- So, if we move a little bit, we still feel approximately the same gravitation.
- Thus, all the components $\dfrac{\partial V}{\partial x_i}$ of the gradient of the gravitational field must be close to 0.
- This is equivalent to requiring that the squares of these derivatives be small.

11. Beyond Netwon's Equations (cont-d)

- Reminder: all the squares $\left(\dfrac{\partial V}{\partial x_i}\right)^2$ are small.
- Small is $\exp(-k\cdot V)$, "and" is product, so

$$\mu(x)=\prod_x \prod_{i=1}^3 \exp\left(-k\cdot\left(\frac{\partial V}{\partial x_i}\right)^2\right).$$

- Here, $\mu=\exp(-k\cdot S)$, and in the limit, $S=\int L\,dx$, where $L(x)\stackrel{\text{def}}{=}\sum_{i=1}^3\left(\dfrac{\partial V}{\partial x_i}\right)^2$.
- It is known that minimizing this expression leads to the equation $\sum_{i=1}^3 \dfrac{\partial^2 V}{\partial x_i^2}=0$.
- This equation leads to Newton's gravitational potential

$$V(x)\sim\frac{1}{r}.$$

12. Discussion

- Similar arguments can lead to other known action principles.
- Thus, similar arguments can lead to other fundamental physical equations.
- At present, this is just a theoretical exercise/proof of concept.
- Its main objective is to provide one more validation for the existing fuzzy methodology:
  - it transforms informal ("fuzzy") description of physical phenomena
  - into well-known physical equations.
- Maybe when new physical phenomena will be discovered, fuzzy methodology may help find the equations?

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14. Appendix: Variational Equations

- *Objective:* $S = \int L(x, \dot{x})\,dt \to \min$.
- Hence, $S(\alpha) = \int L(x + \alpha \cdot \Delta x, \dot{x} + \alpha \cdot \Delta \dot{x})\,dt \to \min$ at $\alpha = 0$.
- So, $\dfrac{\partial S}{\partial \alpha} = \displaystyle\int \left(\frac{\partial L}{\partial x} \cdot \Delta x + \frac{\partial L}{\partial \dot{x}} \cdot \Delta \dot{x} \right) dt = 0$.
- Integrating the second term by parts, we conclude that

$$\int \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \right) \cdot \Delta x\,dt = 0.$$

- This must be true for $\Delta x(t) \approx \delta(t - t_0)$, so

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0.$$

- The resulting equations are known as *Euler-Lagrange equations.*

15. Variational Equations (cont-d)

- *Reminder:* $\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0.$
- In the Newton's case, $L = V(x) - \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2.$
- Here, $\frac{\partial L}{\partial x_i} = \frac{\partial V}{\partial x_i}, \frac{\partial L}{\partial \dot{x}_i} = -m \cdot \frac{dx_i}{dt}$, so Euler-Lagrange's equations take the form $\frac{\partial V}{\partial x} + m \cdot \frac{d}{dt}\left(\frac{dx_i}{dt}\right) = 0.$
- This is equiv. to Newton's equations $m \cdot \frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i}.$
- In the general case, Euler-Lagrange equations take the form $\frac{\partial L}{\partial \varphi} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \varphi_{,i}}\right) = 0$, where $\varphi_{,i} \stackrel{\text{def}}{=} \frac{\partial \varphi}{\partial x_i}.$