Towards Optimal Placement of Bio-Weapon Detectors

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1. Formulation of the Practical Problem

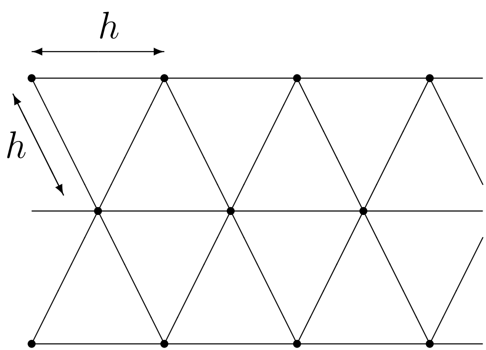
- Biological weapons are difficult and expensive to detect.
- Within a limited budget, we can afford a limited number of bio-weapon detector stations.
- It is therefore important to find the optimal locations for such stations.
- A natural idea is to place more detectors in the areas with more population.
- However, such a commonsense analysis does not tell us how many detectors to place where.
- To decide on the exact detector placement, we must formulate the problem in precise terms.

2. Towards Precise Formulation of the Problem

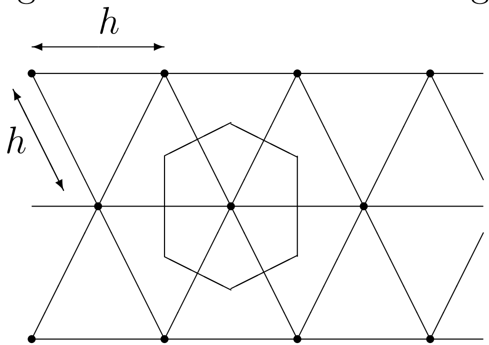
- The adversary's objective is to kill as many people as possible.
- Let $\rho(x)$ be a population density in the vicinity of the location x.
- Let N be the number of detectors that we can afford to place in the given territory.
- Let d_0 be the distance at which a station can detect an outbreak of a disease.
- Often, $d_0 = 0$ – we can only detect a disease when the sources of this disease reach the detecting station.
- We want to find $\rho_d(x)$ – the density of detector placement.
- We know that $\int \rho_d(x)\,dx = N$.

3. Optimal Placement of Sensors

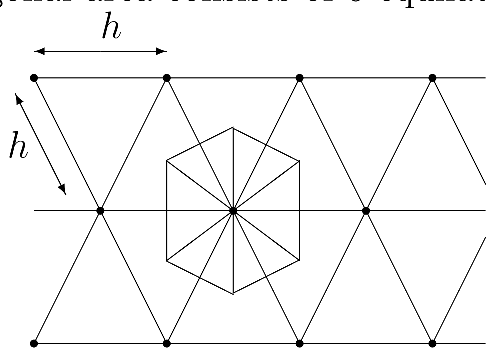
- We want to place the sensors in an area in such a way that
  - the largest distance D to a sensor
  - is as small as possible.
- It is known that the smallest such number is provided by an equilateral triangle grid:



For the equilateral triangle placement, points which are closest to a given detector forms a hexagonal area:



This hexagonal area consists of 6 equilateral triangles:



4. Optimal Placement of Sensors (cont-d)

- In each $\triangle$, the height $h/2$ is related to the side s by the formula $\dfrac{h}{2} = s\cdot\cos(60^\circ) = s\cdot\dfrac{\sqrt{3}}{2}$, hence $s = h\cdot\dfrac{\sqrt{3}}{3}$.
- Thus, the area A_t of each triangle is equal to

$$A_t = \frac{1}{2}\cdot s\cdot\frac{h}{2} = \frac{1}{2}\cdot\frac{\sqrt{3}}{3}\cdot\frac{1}{2}\cdot h^2 = \frac{\sqrt{3}}{12}\cdot h^2.$$

- So, the area A_s of the whole set is equal to 6 times the triangle area: $A_s = 6\cdot A_t = \dfrac{\sqrt{3}}{2}\cdot h^2$.
- In a region of area A, there are $A\cdot\rho_d(x)$ sensors, they cover area $(A\cdot\rho_d(x))\cdot A_s$.
- The condition $A = (A\cdot\rho_d(x))\cdot A_s = (A\cdot\rho_d(x))\cdot\dfrac{\sqrt{3}}{2}\cdot h^2$ implies that $h = \dfrac{c_0}{\sqrt{\rho_d(x)}}$, with $c_0 \stackrel{\text{def}}{=} \sqrt{\dfrac{2}{\sqrt{3}}}$.

5. Estimating the Effect of Sensor Placement

- The adversary places the bio-weapon at a location which is the farthest away from the detectors.
- This way, it will take the longest time to be detected.
- For the grid placement, this location is at one of the vertices of the hexagonal zone.
- At these vertices, the distance from each neighboring detector is equal to $s = h \cdot \dfrac{\sqrt{3}}{3}$.
- By know that $h = \dfrac{c_0}{\sqrt{\rho_d(x)}}$, so $s = \dfrac{c_1}{\sqrt{\rho_d(x)}}$, with

$$c_1 = \frac{\sqrt{3}}{3} \cdot c_0 = \frac{\sqrt[4]{3} \cdot \sqrt{2}}{3}.$$

- Once the bio-weapon is placed, it starts spreading until it reaches the distance d_0 from the detector.

6. Effect of Sensor Placement (cont-d)

- The bio-weapon is placed at a distance $s = \dfrac{c_1}{\sqrt{\rho_d(x)}}$ from the nearest sensor.
- Once the bio-weapon is placed, it starts spreading until it reaches the distance d_0 from the detector.
- In other words, it spreads for the distance $s - d_0$.
- During this spread, the disease covers the circle of radius $s - d_0$ and area $\pi \cdot (s - d_0)^2$.
- The number of affected people $n(x)$ is equal to:

$$n(x) = \pi \cdot (s - d_0)^2 \cdot \rho(x) = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x).$$

7. Precise Formulation of the Problem

- For each location x, the number of affected people $n(x)$ is equal to:

$$n(x) = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x).$$

- The adversary will select a location x for which this number $n(x)$ is the largest possible:

$$n = \max_x \left(\pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x) \right).$$

- Resulting problem:
  - given population density $\rho(x)$, detection distance d_0, and number of sensors N,
  - find a function $\rho_d(x)$ that minimizes the above expression n under the constraint $\int \rho_d(x)\,dx = N$.

8. Main Lemma

- *Reminder:* we want to minimize the worst-case damage $n = \max\limits_x n(x)$.
- *Lemma:* for the optimal sensor selection, $n(x) = \text{const}$.
- *Proof by contradiction:* let $n(x) < n$ for some x; then:
  - we can slightly increase the detector density at the locations where $n(x) = n$,
  - at the expense of slightly decreasing the location density at locations where $n(x) < n$;
  - as a result, the overall maximum $n = \max\limits_x n(x)$ will decrease;
  - but we assumed that n is the smallest possible.
- *Thus:* $n(x) = \text{const}$; let us denote this constant by n_0.

9. Towards the Solution of the Problem

- We have proved that $n(x) = \text{const} = n_0$, i.e., that

$$n_0 = \pi \cdot \left(\frac{c_1}{\sqrt{\rho_d(x)}} - d_0 \right)^2 \cdot \rho(x).$$

- Straightforward algebraic transformations lead to:

$$\rho_d(x) = \frac{2 \cdot \sqrt{3}}{9} \cdot \frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}} \right)^2}.$$

- The value c_2 must be determined from the equation

$$\int \rho_d(x)\, dx = N.$$

- Thus, we arrive at the following solution.

10. Solution

- *General case:* the optimal detector location is characterized by the detector density

$$\rho_d(x) = \frac{2 \cdot \sqrt{3}}{9} \cdot \frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}}\right)^2}.$$

- Here the parameter c_2 must be determined from the equation $\int \frac{2 \cdot \sqrt{3}}{9} \cdot \frac{1}{\left(d_0 + \frac{c_2}{\sqrt{\rho(x)}}\right)^2} \, dx = N.$

- *Case of $d_0 = 0$:* in this case, the formula for $\rho_d(x)$ takes a simplified form $\rho_d(x) = C \cdot \rho(x)$ for some constant C.

- In this case, from the constraint, we get:

$$\rho_d(x) = \frac{N}{N_p} \cdot \rho(x), \text{ where } N_p \text{ is the total population.}$$

11. Towards More Relevant Objective Functions

- We assumed that the adversary wants to maximize the number $\int \rho(x)\,dx$ of people affected by the bio-weapon.
- The actual adversary's objective function may differ from this simplified objective function.
- For example, the adversary may take into account that different locations have different publicity potential.
- In this case, the adversary maximizes the weighted value $\int_A \widetilde{\rho}(x)\,dx$, where $\widetilde{\rho}(x) \stackrel{\text{def}}{=} w(x) \cdot \rho(x)$.
- Here, $w(x)$ is the importance of the location x.
- From the math. viewpoint, the problem is the same – w/ "effective population density" $\widetilde{\rho}(x)$ instead of $\rho(x)$.
- Thus, if we know $w(x)$, we can find the optimal detector density $\rho_d(x)$ from the above formulas.

12. Fuzzy Techniques May Help

- If we know the importance values $w(x)$ *exactly*, then we can find the optimal sensor placement $\rho_d(x)$.
- *Problem:* we usually only have expert estimates for $w(x)$.
- These estimates are often formulated in terms of words form *natural language* like "small".
- To formalize these estimates, we can use *fuzzy* techniques and get fuzzy estimates for $w(x)$.
- Once we have the fuzzy values of $w(x)$, we compute *fuzzy recommendations* for the detector density $\rho_d(x)$.
- *How:* we can use Zadeh's extension principle.