Mamdani Approach to Fuzzy Control, Logical Approach, What Else?

Samuel Bravo and Jaime Nava
Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
sbravo09@gmail.com
jenava@miners.utep.edu

1. Need for Fuzzy Control

- In many application areas,
  - we do not have the exact control strategies, but
  - we have human operators who are skilled in the corresponding control.
- Human operators are often unable to describe their knowledge in a precise quantitative form.
- Instead, they describe their knowledge by using words from natural language.
- These rules usually have the form "If $A_i(x)$ then $B_i(u)$", where x is the input and u is the resulting control.
- For example, a rule may say "If a car in front is somewhat too close, break a little bit".
- Fuzzy control is a set of techniques for transforming these rules into a precise control strategy.

2. Mamdani Approach to Fuzzy Control: Historically the First

- For a given input x, a control value u is reasonable if:
  - the 1st rule is applicable, i.e., its condition $A_1(x)$ is satisfied and its conclusion $B_1(u)$ is satisfied,
  - or the 2nd rule is applicable, i.e., its condition $A_2(x)$ is satisfied and its conclusion $B_2(u)$ is satisfied,
  - etc.
- Thus, the condition $R(x, u)$ "the control u is reasonable for the input x" takes the form

$$(A_1(x) \,\&\, B_1(u)) \vee (A_2(x) \,\&\, B_2(u)) \vee \ldots$$

- To get control value $u(x_0)$, we apply a defuzzification procedure to the corr. membership function $R(x_0, u)$.

3. Logical (More Recent) Approach to Fuzzy Control

- *Main idea:* simply state that all the rules are valid, i.e., that the following statement holds:

$$(A_1(x) \to B_1(u))\,\&\,(A_2(x) \to B_2(u))\,\&\,\ldots$$

- For example, we can interpret $A \to B$ as $\neg A \vee B$, in which case the above formula has the form

$$(\neg A_1(x) \vee B_1(u))\,\&\,(\neg A_2(x) \vee B_2(u))\,\&\,\ldots$$

- Equivalently, we can use the form

$$(A'_1(x) \vee B_1(u))\,\&\,(A'_2(x) \vee B_2(u))\,\&\,\ldots,$$

where $A'_i(x)$ denotes $\neg A_i(x)$.

4. Both Approaches Have a Universality Property

- *Fact:* both
  - Mamdani's approach to fuzzy control and
  - logical approach to fuzzy control

  have a universality (universal approximation) property.
- *Meaning of universal approximation property:*
  - an arbitrary control strategy can be,
  - with arbitrary accuracy,
  - approximated by controls generated by this approach.

5. Corresponding Crisp Universality Property

- *Why* do the corresponding fuzzy controls have the universal approximation property?
- *Intuitive explanation:* because the corresponding crisp formulas have the universal property.
- *In precise terms:* for finite sets X and U, any relation $C(x,u)$ on $X \times U$ can be represented in both forms

$$(A_1(x) \,\&\, B_1(u)) \vee (A_2(x) \,\&\, B_2(u)) \vee \ldots;$$

$$(A_1(x) \to B_1(u)) \,\&\, (A_2(x) \to B_2(u)) \,\&\, \ldots.$$

- *Proof:* an arbitrary crisp property $C(x,u)$ is described by the set $C = \{(x,u) : C(x,u)\}$, so:

$$C(x,u) \Leftrightarrow \vee_{(x_0,u_0)\in C}((x = x_0) \,\&\, (u = u_0));$$

$$C(x,u) \Leftrightarrow \&_{(x_0,u_0)\notin C}((x = x_0) \to (u \neq u_0)).$$

- *Fact:* the corr. CNF & DNF representations are actively used in digital design; e.g., in vending machines.

6. Fuzzy Control: What Other Approaches Are Possible?

- Both Mamdani's and logical approaches are actively used in fuzzy control.
- The fact that both approaches are actively used means that both have advantages and disadvantages.
- In other words, this means that none of these two approaches is perfect.
- Since both approaches are not perfect, it is reasonable to analyze what other approaches are possible.
- In this paper, we start this analysis by analyzing what type of crisp forms like

$$(A_1(x)\,\&\,B_1(u)) \vee (A_2(x)\,\&\,B_2(u)) \vee \ldots;$$

$$(A_1(x) \to B_1(u))\,\&\,(A_2(x) \to B_2(u))\,\&\,\ldots.$$

are possible.

7. Definitions

- *By a* binary operation, *we mean a function* $f : \{0,1\} \times \{0,1\} \to \{0,1\}$ *that transforms Boolean values.*
- *A pair of binary operations* $(\odot, \ominus)$ *s.t.* $\ominus$ *is commutative and associative has a* universality property *if:*
  - *for every two finite sets* X *and* Y,
  - *an arbitrary relation* $C(x,u)$ *can be represented, for some* $A_i(x)$ *and* $B_i(u)$, *as*

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \ldots$$

- *We say that pairs* $(\odot, \ominus)$ *and* $(\odot', \ominus)$ *are* similar *if the relation* $\odot'$ *has one of the following forms:*

$$a \odot' b \stackrel{\text{def}}{=} \neg a \odot b, \; a \odot' b \stackrel{\text{def}}{=} a \odot \neg b, \; \text{ or } a \odot' b \stackrel{\text{def}}{=} \neg a \odot \neg b.$$

8. Main Result

- **Theorem.** *Every pair of operations with the universality property is similar to one of the following pairs:*

$$(\vee, \&), (\&, \vee), (\oplus, \vee), (\oplus, \&), (\oplus', \vee), (\equiv, \vee), (\equiv, \&).$$

- Thus, in addition to the Mamdani and logical approaches, we have 4 other pairs with the universality property.
- In essence, we have 2 new forms w/ "exclusive or" $\oplus$:

$$(A_1(x) \,\&\, B_1(u)) \oplus (A_2(x) \,\&\, B_2(u)) \oplus \ldots;$$

$$(A_1(x) \vee B_1(u)) \oplus (A_2(x) \vee B_2(u)) \oplus \ldots$$

- The meaning of the new forms: we restrict ourselves to the cases when exactly one rule is applicable.

9. Proof: Main Lemmas

- *If the pairs* $(\odot, \ominus)$ *and* $(\odot', \ominus)$ *are similar, then the following two statements are equivalent to each other:*
  - *the pair* $(\odot, \ominus)$ *has the universality property;*
  - *the pair* $(\odot', \ominus)$ *has the universality property.*
- *Out of all binary operations, only the following six are commutative and associative:*
  - *the "zero" operation s.t.* $f(a, b) = 0$ *for all* a *and* b*;*
  - *the "one" operation s.t.* $f(a, b) = 1$ *for all* a *and* b*;*
  - *the "and" operation s.t.* $f(a, b) = a \,\&\, b$*;*
  - *the "or" operation s.t.* $f(a, b) = a \vee b$*;*
  - *the "exclusive or" operation s.t.* $f(a, b) = a \oplus b$*;*
  - *the operation* $a \oplus' b \stackrel{\text{def}}{=} a \oplus \neg b$*.*

10. Proof: Details

- To describe a binary operation, one needs to describe four Boolean values: $f(0,0)$, $f(0,1)$, $f(1,0)$, and $f(1,1)$.
- Each of these four quantities can have two different values: 0 and 1.
- A natural way to classify these operations is to describe how many 1s we have as values $f(a,b)$: 0, 1, 2, 3, or 4.
- When we have zero 1s, then $\forall a \forall b\, f(a,b) = 0$.
- If we use this operation as $\ominus$, we get a constant 0:

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \ldots = 0$$

- If we use this operation as $\odot$, we get a constant independent on x and u:

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \ldots = 0 \ominus 0 \ominus \ldots$$

- In both cases, we cannot have universality property.

11. Proof (cont-d)

- When we have four 1s, this means that $f(a, b) = 1$ for all a and b.
- In this case, we can similarly prove that we have no universality property.
- When we have a single one, this means that we have an operation similar to "and":

$$a \,\&\, b, \quad a \,\&\, \neg b, \quad \neg a \,\&\, b, \quad \neg a \,\&\, \neg b.$$

- Similarly, we can prove that when we have three ones, this means that we have an operation similar to "or".
- For two 1s, we have $\binom{4}{2} = 6$ options:

$$f(a, b) = a, f(a, b) = \neg a, f(a, b) = b, f(a, b) = \neg b, a \oplus b, a \oplus \neg b.$$

- By analyzing these operations one by one, we describe all commutative and associative operations.

12. Proof (last part)

- Due to the above result, we only need to consider the above six operations $\ominus$: 0, 1, &, $\vee$, $\oplus$, and $\equiv$.
- We have already shown that 0 and 1 do not have universality property, and that $\equiv$ is equivalent to $\oplus$.
- So, it is sufficient to consider $\ominus =$ &, $\vee$, and $\oplus$.
- For each of these $\ominus$ operations, we consider all possible $\odot$ operations.
- It is enough to consider one $\odot$ operation from each equivalence class.
- So, we take $\odot = 0$, 1, &, $\vee$, $\oplus$, and cases when $f(a, b)$ depends only on one of the variables a or b.
- By analyzing these cases one by one, we exclude all the pairs except for the one listed in the theorem.