Wiener's Conjecture About Transformation Groups Helps Predict Which Fuzzy Techniques Work Better

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1. Formulation of the Problem

- Often, application succeeds only when we select specific fuzzy techniques (t-norm, membership f-n, etc.).
- In different applications, different techniques are the best.
- How to find the best technique?
- Exhaustive search of all techniques is not an option: there are too many of them.
- We need to come up with a narrow class of promising techniques, so that trying them all is realistic.
- We show that transformation groups – motivated by N. Wiener's conjecture – lead to such a narrowing.
- This conjecture was, in its turn, motivated by observations about human vision.

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2. Wiener's Conjecture: Reminder

- The closer we are to an object, the better we can determine its shape.
- Experiments show that there are distinct phases in this determination.
- When the object is very far, all we see is a formless blurb.
- In other words, objects obtained from other by arbitrary smooth transformations cannot be distinguished.
- When the object gets closer, we can detect whether it is smooth or has sharp angles.
- We may see a circle as an ellipse, a square as a rhombus (diamond).
- At this stage, images obtained by a projective transformation are indistinguishable.

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3. Wiener's Conjecture (cont-d)

- When the object gets closer, we can detect which lines are parallel but we may not yet detect the angles.
- For example, we are not sure whether what we see is a rectangle or a parallelogram.
- This stage corresponds to affine transformation.
- Then, we have a stage of similarity transformations – when we detect the shape but cannot yet detect its size.
- Finally, when the object is close enough, we can detect both its shape and its size.
- Each stage can be this described by an appropriate transformation group (see a formal description below).

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4. Wiener's Conjecture: Result

- Humans result from billions of years of evolution. So, Wiener conjectured that:
  - if there was a group intermediate between, e.g., all projective and all continuous transformations,
  - our vision mechanism would have used it.
- Thus, according to the 1940s Wiener's conjecture, such intermediate groups are not possible.
- In the 1960s, Wiener's conjecture was proven.
- In the 1-D case, projective transformations are simply fractionally linear, and affine are simply linear.
- Thus, any group containing all 1-D linear transformation is:
  - either the group of all fractionally-linear transf.
  - or the group of all transformations.

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5. How Wiener's Conjecture Helps: General Idea

- Fuzzy degrees are not uniquely determined.
- Different elicitation techniques lead, in general, to different values.
- Sometimes, different scales are related by a linear transformation, sometimes by a non-linear one.
- In practice, we want a description with finitely many parameters.
- Thus, we want a finite-dimensional transformation group.
- Due to the above result, all such transformations are fractionally linear.
- We show that this can explain why some t-norms, membership functions, etc., are empirically more successful.

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6. Different Assignment Procedures Are In Use

- Intelligent systems use several different procedures for assigning numeric values that describe uncertainty.
- The same expert's degree of uncertainty that he expresses, e.g., by the expression "for sure", can lead:
  - to 0.9 if we apply one procedure, and
  - to 0.8 if another procedure is used.
- 1 foot and 12 inches describe the same length, but in different scales.
- We can say that 0.9 and 0.8 represent the same degree of certainty in two different *scales*.
- Some scales are different even in the fact that they use an interval different from $[0, 1]$ to represent uncertainty.
- For example, the famous MYCIN system uses the interval $[-1, 1]$.

7. Transformations Between Reasonable Scales

- Let F denote the class of reasonable transformations of degrees of uncertainty. If:
  - a function $x \to f(x)$ is a reasonable transformation from a scale A to some scale B, and
  - a function $y \to g(y)$ is a reasonable transformation from B into some other scale C,
  - then the transformation $x \to g(f(x))$ from A to C is also reasonable.
- In other words, the class F of all reasonable transformations must be closed under composition. Also:
  - if $x \to f(x)$ is a reasonable transformation from a scale A to scale B,
  - then the inverse function is a reasonable transformation from B to A.
- Thus, F must be a *transformation group*.

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8. Examples of Reasonable Transformations

- A natural method to assign a truth value $t(S)$ to a statement S is to ask several experts and take
$$t(S) = \frac{N(S)}{N}.$$
- The more expert we ask, the more reliable is this estimate.
- However, in the presence of Nobelists, experts may say nothing or follow the majority.
- After we add M experts who do not answer anything and M' who follow the majority, we get
$$t' = \frac{N(S) + M'}{N + M + M'} = \frac{N \cdot t + M'}{N + M + M'} = a \cdot t + b.$$
- The transformation from an old scale $t(S)$ to a new scale t' is a linear function.

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9. Definition and Main Result

- By a *rescaling* we mean a strictly increasing continuous function f that is defined on an interval $[a,b] \subseteq \mathbb{R}$.
- Suppose a set F of rescalings is a connected Lie group which contains, for all $N, M, M' \ge 0$, a transformation

$$t \to \frac{N \cdot t + M'}{N + M + M'}.$$

- Elements of this set F will be called *reasonable transformations.*
- *Result:* Every reasonable transformation $f(x)$ is fractionally linear: $f(x) = \dfrac{a \cdot x + b}{c \cdot x + d}$.

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10. Normalizations

- To compare degrees in different scales, we need to "normalize" them.
- Often, there exists an alternative a for which we are absolute sure that it is not possible: $\mu(a) = 0$.
- It is natural to require that this value 0 should remain the same after the "normalization" transformation.
- By a *normalization* we mean a reasonable transformation $f(x)$, for which $f(0) = 0$.
- *Result:* Every normalization has the form $f(x) = \dfrac{k \cdot x}{1 + d \cdot x}$.
- *Comment.* This class includes the most widely used linear normalization $\mu'(x) = \dfrac{\mu(x)}{\max\limits_y \mu(y)}$.

11. Selecting Membership Functions

- Suppose that we have a fuzzy notion like "small".
- For $x = 0$, we are sure that it is small.
- Until we reach large values, the bigger x, the less we are certain that x is small.
- There are thus two ways to represent our uncertainty:
  - we can use the value of a membership function $\mu(x)$;
  - we can also use the value x itself – since the larger x, the larger our uncertainty.
- The transformation between these scales must be reasonable.
- So, a membership function must be piecewise fractionally linear.
- Triangular and trapezoid functions – most efficient – are indeed examples of such functions.

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12. "And"-Operations

- When we communicate, we often make implicit assumptions.
- For example, when we ask a doctor to estimate the efficiency of a certain treatment t:
  - the doctor may interpret it as estimating the proportion of patients who gets well,
  - or as proportion of patients who got well because of t – and not by itself.
- In other words, we estimate either $d(W)$ or $d(W \,\&\, T)$, where T means that the treatment worked.
- It makes sense to require that the transformation $d(W) \to d(W \,\&\, d(T))$ is reasonable.
- In fuzzy logic, we estimate $d(W \,\&\, T)$ as $f_\&(d(W), d(T))$.
- So, we require that $a \to f_\&(a, b)$ is reasonable for all b.

13. "And"-Operations (T-Norms), "Or"-Operations (T-Conorms)

- *Reminder:* we require that $a \to f_\&(a,b)$ is reasonable for all b.

- *Result:* all such t-norms are either $f_\&(a,b) = \min(a,b)$ or
$$f_\&(a,b) = \frac{a \cdot b}{k + (1-k) \cdot (a + b - a \cdot b)}.$$

- For t-conorms ("'or"-operations), we similarly get $f_\vee(a,b) = \max(a,b)$ or
$$f_\vee(a,b) = \frac{a + b + (k-1) \cdot a \cdot b}{1 + k \cdot a \cdot b}.$$

- Most widely used min, max, $a \cdot b$, and $a + b - a \cdot b$ are indeed examples of such operations.

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14. Negation Operations etc.

- A negation operation can be defined as a function $f_\neg(x)$ which extends the usual negation from $\{0, 1\}$ to $[0, 1]$:

$$f_\neg(0) = 1 \text{ and } f_\neg(1) = 0.$$

- We can express our uncertainty in a statement A:
  - either by our degree of belief $d(A)$ in A,
  - or by our degree of belief $d(\neg A) = f_\neg(d(A))$ in $\neg A$.
- The transformation $f_\neg(x) : d(A) \to d(\neg A)$ is reasonable, so $f_\neg(x) = \dfrac{1-x}{1+k\cdot x}$.
- For $k = 0$, we get the original negation $f_\neg(x) = 1 - x$.
- For $k \neq 0$, we get Sugeno operations which are known to be a good fit for human reasoning.
- Similarly, we explain which defuzzification to use, why sigmoid activation functions are efficient, etc.

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15. Acknowledgment

This work was supported in part by the National Science Foundation grants:

- HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and
- DUE-0926721.

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