Formalizing the Informal, Precisiating the Imprecise: How Fuzzy Logic Can Help Mathematicians and Physicists by Formalizing Their Intuitive Ideas

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1. Outline

- Fuzzy methodology:
  - transforms expert ideas – formulated in terms of words from natural language,
  - into precise rules and formulas.
- In this talk, we show that by applying this methodology to intuitive physical and mathematical ideas:
  - we can get known fundamental physical equations and
  - we can get known mathematical techniques for solving these equations.
- This makes us confident that in the future, fuzzy techniques will still help physicists and mathematicians.

2. Fuzzy Is Most Successful When We Have Partial Knowledge

- Fuzzy methodology has been invented to transform:
  - expert ideas – formulated in terms of words from natural language,
  - into precise rules and formulas, rules and formulas understandable by a computer.
- Fuzzy methodology has led to many successful applications, especially in intelligent control.
- Major successes of fuzzy methodology is when we only have *partial* knowledge.
- This is true for all known fuzzy control success stories: washing machines, camcoders, elevators, trains, etc.

3. Is Fuzzy Poor Man Data Processing?

- From this viewpoint:
  - as we gain more knowledge about a system,
  - a moment comes when we do not need to use fuzzy techniques any longer.
  - we will be able to use traditional (crisp) techniques.
- So, fuzzy techniques look like a (successful but still) *intermediate* step,
  - "poor man's" data processing techniques,
  - that need to be used only if we cannot apply "more optimal" traditional methods.
- We show, on example of the study of physical world, that fuzzy methodology can be very useful beyond that.

4. Studying Physical World

- When we study the physical world, our first task is *physical*: to find the *physical laws*.
- In precise terms, these are *equations* that describe how the values of physical quantities change with time.
- Once we have found these equations, the next task is *mathematical*:
  - we need to solve these equations
  - to predict the future values of physical quantities.
- Both tasks are not easy. In both tasks, we:
  - start with informal ideas, and
  - gradually move to exact equations and exact algorithms for solving these equations.
- But such *precisiation* of informal ideas is exactly what fuzzy techniques were invented for, so let's use them.

5. Newton's Physics: Informal Description

- A body usually tries to go to the points x where its potential energy $V(x)$ is the smallest.
- For example, a moving rock on the mountain tries to go down.
- The sum of the potential energy $V(x)$ and the kinetic energy K is preserved:

$$K = \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt} \right)^2 .$$

- Thus, when the body minimizes its potential energy, it thus tries to maximize its kinetic energy.
- We will show that when we apply the fuzzy techniques to this informal description, we get Newton's equations

$$m \cdot \frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i}.$$

6. First Step: Selecting a Membership Function

- The body tries to get to the areas where the potential energy $V(x)$ is small.
- We need to select the corresponding membership function $\mu(V)$.
- For example, we can poll several (n) experts and if $n(V)$ of them consider V small, take $\mu(V) = \dfrac{n(V)}{n}$.
- In physics, we only know *relative* potential energy – relative to some level.
- If we change that level by V_0, we replace V by $V + V_0$.
- So, values V and $V + V_0$ represent the same value of the potential energy – but for different levels.
- A seemingly natural formalization: $\mu(V) = \mu(V + V_0)$.
- Problem: we get useless $\mu(V) = \text{const}$.

7. Re-Analyzing the Polling Method

- In the poll, the more people we ask, the more accurate is the resulting opinion.
- Thus, to improve the accuracy of the poll, we add m folks to the original n top experts.
- These m extra folks may be too intimidated by the original experts.
- With the new experts mute, we still have the same number $n(V)$ of experts who say "yes".
- As a result, instead of the original value $\mu(V)=\dfrac{n(V)}{n}$, we get $\mu'(V)=\dfrac{n(V)}{n+m}=c\cdot\mu(V)$, where $c=\dfrac{n}{n+m}$.
- These two membership functions $\mu(V)$ and $\mu'(V)=c\cdot\mu(V)$ represent the same expert opinion.

8. Resulting Formalization of the Physical Intuition

- *How* to describe that potential energy is small?
- *Idea:* value V and $V + V_0$ are equivalent – they differ by a starting level for measuring potential energy.
- *Conclusion:* membership functions $\mu(V)$ and $\mu(V+V_0)$ should be equivalent.
- *We know:* membership functions $\mu(V)$ and $\mu'(V)$ are equivalent if $\mu'(V) = c \cdot \mu(V)$.
- *Hence:* for every V_0, there is a value $c(V_0)$ for which

$$\mu(V + V_0) = c(V_0) \cdot \mu(V).$$

- *It is known* that the only monotonic solution to this equation is $\mu(V) = a \cdot \exp(-k \cdot V)$.
- *So* we will use this membership function to describe that the potential energy is small.

9. Resulting Formalization of the Physical Intuition (cont-d)

- *Reminder:* we use $\mu(V) = a \cdot \exp(-k \cdot V)$ to describe that potential energy is small.
- *How* to describe that kinetic energy is large?
- *Idea:* K is large if $-K$ is small.
- *Resulting membership function:*

$$\mu(K) = \exp(-k \cdot (-K)) = \exp(k \cdot K).$$

- *We want* to describe the intuition that
  - the potential energy is small *and*
  - that the kinetic energy is large *and*
  - that the same is true at different moments of time.
- According to fuzzy methodology, we must therefore select an appropriate "and"-operation (t-norm) $f_{\&}(a, b)$.

10. How to Select an Appropriate t-Norm

- In principle, if we have two completely independent systems, we can consider them as a single system.
- Since these systems do not interact with each other, the total energy E is simply equal to $E_1 + E_2$.
- We can estimate the smallness of the total energy in two different ways:
  - we can state that the total energy $E = E_1 + E_2$ is small: certainty $\mu(E_1 + E_2)$, or
  - we can state that both E_1 *and* E_2 are small:

$$f_{\&}(\mu(E_1), \mu(E_2)).$$

- It is reasonable to require that these two estimates coincide: $\mu(E_1 + E_2) = f_{\&}(\mu(E_1), \mu(E_2))$.
- This requirement enables us to uniquely determine the corresponding t-norm: $f_{\&}(a_1, a_2) = a_1 \cdot a_2$.

11. Resulting Model

- *Idea:* at all moments of time $t_1, \ldots, t_N$, the potential energy V is small, and the kinetic energy K is large.
- Small is $\exp(-k \cdot V)$, large is $\exp(k \cdot K)$, "and" is product, thus the degree $\mu(x(t))$ is

$$\mu(x(t)) = \prod_{i=1}^{N} \exp(-k \cdot V(t_i)) \cdot \prod_{i=1}^{N} \exp(k \cdot K(t_i)).$$

- So, $\mu(x(t)) = \exp(-k \cdot S)$, w/ $S \stackrel{\text{def}}{=} \sum_{i=1}^{N} (V(t_i) - K(t_i))$.
- In the limit $t_{i+1} - t_i \to 0$, $S \to \int (V(t) - K(t))\,dt$.
- The most reasonable trajectory is the one for which $\mu(x(t)) \to \max$, i.e., $S = \int L\,dt \to \min$, where

$$L \stackrel{\text{def}}{=} V(t) - K(t) = V(t) - \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2.$$

12. This Model Leads to Newton's Equations

- *Reminder:* $S = \int L\,dt \to \min$, where

$$L \stackrel{\text{def}}{=} V(t) - K(t) = V(t) - \frac{1}{2} \cdot m \cdot \sum_{i=1}^{3} \left(\frac{dx_i}{dt} \right)^2.$$

- Most physical laws are now formulated in terms of the *Principle of Least Action* $S = \int L\,dt \to \min$.
- E.g., for the above L, we get Newtonian physics.
- So, *fuzzy indeed implies Newton's equations.*
- *Newton's physics:* only one trajectory, with $S \to \min$.
- With the *fuzzy* approach, we also get the degree $\exp(-k \cdot S)$ w/which other trajectories are reasonable.
- In *quantum* physics, each non-Newtonian trajectory is possible with "amplitude" $\exp(-k \cdot S)$ (for complex k).
- This makes the above derivation even more interesting.

13. Beyond the Simplest Netwon's Equations

- In our analysis, we assume that the expression for the potential energy field $V(x)$ is given.
- In reality, we must also find the equations that describe the corresponding field.
- Simplest case: gravitational field.
- The gravitational pull of the Earth is caused by the Earth as a whole.
- So, if we move a little bit, we still feel approximately the same gravitation.
- Thus, all the components $\dfrac{\partial V}{\partial x_i}$ of the gradient of the gravitational field must be close to 0.
- This is equivalent to requiring that the squares of these derivatives be small.

14. Beyond Netwon's Equations (cont-d)

- Reminder: all the squares $\left(\dfrac{\partial V}{\partial x_i}\right)^2$ are small.
- Small is $\exp(-k\cdot V)$, "and" is product, so

$$\mu(x)=\prod_x \prod_{i=1}^3 \exp\left(-k\cdot\left(\frac{\partial V}{\partial x_i}\right)^2\right).$$

- Here, $\mu=\exp(-k\cdot S)$, and in the limit, $S=\int L\,dx$, where $L(x)\stackrel{\text{def}}{=}\sum_{i=1}^3\left(\dfrac{\partial V}{\partial x_i}\right)^2$.
- It is known that minimizing this expression leads to the equation $\sum_{i=1}^3 \dfrac{\partial^2 V}{\partial x_i^2}=0.$
- This equation leads to Newton's gravitational potential

$$V(x)\sim\frac{1}{r}.$$

15. Discussion

- Similar arguments can lead to other known action principles.
- Thus, similar arguments can lead to other fundamental physical equations.
- At present, this is just a theoretical exercise/proof of concept.
- Its main objective is to provide one more validation for the existing fuzzy methodology:
  - it transforms informal ("fuzzy") description of physical phenomena
  - into well-known physical equations.
- Maybe when new physical phenomena will be discovered, fuzzy methodology may help find the equations?

16. From Equations to Solutions

- The ultimate goal is to predict the future values of the corresponding physical quantities.
- The first step is to find the equations that describe the dynamics of the corresponding particles and/or fields.
- We have shown that fuzzy techniques can help in determining these equations.
- To predict future values, we now need to solve these equations.
- The equations are often complex, and in many situations, no analytical solution is known.
- So, we have to consider approximate methods.
- How can we do it?

17. Idea: Our Knowledge Is Usually Incremental

- At any given moment of time, we have a model which is a reasonably good approximation to reality.
- Then, we find a new, more accurate model:
  - the ideas behind the new model may be revolutionary (e.g., quantum physics, relativity theory),
  - but in terms of predictions, the new theories usually provide a small adjustment to the previous one.
- For example, General Relativity better describes the bending of light near the Sun: by 1.75 arc-seconds.
- Usually, by the time new complex equations appear, we already know how to solve previous equations.
- Thus, the solution x_0 to the previous equations is a first approximation to the solution x of the new equations.

18. How This Idea Is Used

- The difference $x - x_0$ between the old and new solutions can be characterized by some small parameter q.
- The old solution x_0 corresponds to $q = 0$.
- To get a better approximation, we can take into account terms which are linear, quadratic, etc., in q:

$$x = \sum_{i=0}^{\infty} q^i \cdot x_i = x_0 + q \cdot x_1 + q^2 \cdot x_2 + \ldots$$

- In practice, we compute the first few terms in this sum $s_k \stackrel{\text{def}}{=} \sum_{i=0}^{k} q^i \cdot x_i$.
- The first ignored term $q^{k+1} \cdot x_{k+1}$ provides a reasonably accurate description of the approximation error.
- This method often works well, e.g., in celestial mechanics.

19. Divergence: A Problem

- In some other cases, e.g., in quantum electrodynamics, this method only works for small k:
  - we get a good approximation s_0;
  - we get a more accurate approximation s_1;
  - we get an even more accurate approximation s_2;
  - ...
  - until we reach a certain threshold k_0;
  - once this threshold is reached, the approximation accuracy decreases.
- In other words, the series diverge.
- In quantum electrodynamics, the series diverge starting with $k_0 = 137$.
- This divergence is one of the main obstacles to quantum field theory.

20. Maybe Fuzzy Techniques Can Help

- Divergence is largely a theoretical problem.
- In practice, physicists use semi-heuristic methods to come up with meaningful predictions.
- Formalizing imprecise semi-heuristic ideas is one of the main reasons why fuzzy techniques were invented.
- Let us therefore try to use fuzzy techniques to formalize the physicists' reasoning.

21. How Physicists Use Divergent Series

- Physicists usually consider only the approximations until the remaining term $s_{k+1} - s_k$ starts increasing:

$$s_{k+1} - s_k \ll s_k - s_{k-1} \text{ and } s_{k+1} - s_k \ll s_{k+2} - s_{k+1}.$$

- We show that fuzzy logic allows us to come up with a mathematically rigorous formalization of this idea.
- For every k, $x \approx s_k$ with an accuracy proportional to the first ignored term $s_{k+1} - s_k$:

$$x \approx s_k \text{ with accuracy } s_{k+1} - s_k.$$

22. How to Describe the Degree $\mu(x, a, \sigma)$ to Which $x \approx a$, With Accuracy of Order σ"

- *Mathematical ideas:*
  - this degree should be equal to 1 when $x = a$;
  - it should strictly decrease to 0 as x increase up from a;
  - it should strictly decrease to 0 as x decreases down from a.
- *Physical ideas:*
  - we want to apply this function to values of physical quantities;
  - the numerical value of a physical quantity depends:
    * on the choice of a measuring unit and
    * on the choice of a starting point;
  - it is reasonable to require that the degree $\mu(x, a, \sigma)$ should not change of we make a different choice.

23. Scale Invariance

- If we replace a measuring unit by a new unit which is λ times smaller, we get $x \to \lambda \cdot x$.
- For example, $x = 2$ m becomes $x' = 200$ cm.
- Since accuracy is measured in the same units, in the new units, we have $\sigma' = \lambda \cdot \sigma$.
- So, invariance means that for every $\lambda > 0$, we have

$$\mu(\lambda \cdot x, \lambda \cdot a, \lambda \cdot \sigma) = \mu(x, a, \sigma).$$

- Sometimes, the sign of a physical quantity is also arbitrary, so it can change $x \to -x$.
- For example, the direction of a spatial coordinate is a pure convention.
- Accuracy σ describes the *absolute value* $|x - a|$ of the difference $x-a$, so $\sigma' = \sigma$ and $\mu(-x, -a, \sigma) = \mu(x, a, \sigma)$.

24. Shift Invariance and Combination Property

- If we replace the starting point with a new one which is x_0 units lower, we get $x \to x + x_0$.
- The accuracy $\sigma \approx |x - a|$ does not change, so $\mu(x, a, \sigma) = \mu(x + x_0, a + x_0, \sigma)$.
- Often, we have several estimates of this type.
- We should be able to combine them into a single estimate:
  - for every finite set of values a_i and σ_i,
  - we should describe the "and"-combination of all the rules of these types by a single rule of a similar type.
- We have already argued that algebraic product is a good way to formalize "and".

25. Proposition

- Let $\mu(x, a, \sigma)$ be a $[0, 1]$-valued continuous function s.t.:
  - $\mu(a, a, \sigma) = 1$;
  - $\mu(x, a, \sigma)$ strictly decreases for $x \geq a$, strictly increases for $x \leq a$, and tends to 0 as $x \to \pm\infty$;
  - $\mu(\lambda \cdot x, \lambda \cdot a, \lambda \cdot \sigma) = \mu(x, a, \sigma)$;
  - $\mu(-x, -a, \sigma) = \mu(x, a, \sigma)$;
  - $\mu(x + x_0, a + x_0, \sigma) = \mu(x, a, \sigma)$;
  - for every $a_1, \ldots, a_n$, $\sigma_1, \ldots, \sigma_n$, there exist values a, σ, and C for which, for all x, we have

$$\mu(x, a_1, \sigma_1) \cdot \ldots \cdot \mu(x, a_n, \sigma_n) = C \cdot \mu(x, a, \sigma).$$

- Then, $\mu(x, a, \sigma) = \exp\left(-\beta \cdot \left(\dfrac{x-a}{\sigma}\right)^2\right)$ for some β.

26. Back to Our Problem

- The degree to which the rule "$x \approx s_k$ with accuracy $s_{k+1} - s_k$" is satisfied is $\exp\left(-\beta \cdot \dfrac{(x - s_k)^2}{(s_{k+1} - s_k)^2}\right)$.
- The degree to which all these rules are satisfied is equal to the product.
- We select the most probable value x; maximizing the product, we get

$$X_N = \frac{\sum\limits_{k=0}^{N} s_k \cdot (s_{k+1} - s_k)^{-2}}{\sum\limits_{k=0}^{N} (s_{k+1} - s_k)^{-2}}.$$

27. Back to Our Problem (cont-d)

- The actual solution corresponds to $N \to \infty$:

$$x = \lim_{N\to\infty} \frac{\sum_{k=0}^{N} s_k \cdot (s_{k+1} - s_k)^{-2}}{\sum_{k=0}^{N} (s_{k+1} - s_k)^{-2}}.$$

- This formula covers both:
  - the case of a convergent series – in which case it coincides with the limit $\lim s_k$, and
  - the case of the divergent series, in which it leads to $x \approx s_{k_0}$.
- We get a similar result in the probabilistic case, when $x \approx s_k$ with Gaussian approximation error with

$$\sigma \sim |s_{k+1} - s_k|.$$

28. Fuzzy and Physics: Promising Future

- The existing fuzzy methodology enables us:
  - to transform informal ("fuzzy") description of physical phenomena
  - into well-known physical equations.
- This makes us confident that in the future:
  - when new physical phenomena will be discovered,
  - fuzzy methodology may help generate the equations describing these phenomena.
- Fuzzy techniques can lead to an explanation of the known heuristic methods for solving physical equations.
- This makes us confident that in the future, similarly fuzzy techniques will help to transform:
  - informal ideas
  - into new successful mathematical techniques.

29. Future Is Fuzzy!

- People often say "the future is fuzzy" meaning that it is difficult to predict the future exactly.
- But, based on what we observed, we can claim that "the future is fuzzy" in a completely different sense:
  - that the future will see more and more applications of fuzzy techniques,
  - including applications to areas like theoretical physics and numerical mathematics,
  - areas where, at present, there are not many applications of fuzzy.
- The future is fuzzy!

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31. Appendix: Variational Equations

- *Objective:* $S = \int L(x, \dot{x})\, dt \to \min$.
- Hence, $S(\alpha) = \int L(x + \alpha \cdot \Delta x, \dot{x} + \alpha \cdot \Delta \dot{x})\, dt \to \min$ at $\alpha = 0$.
- So, $\dfrac{\partial S}{\partial \alpha} = \displaystyle\int \left(\frac{\partial L}{\partial x} \cdot \Delta x + \frac{\partial L}{\partial \dot{x}} \cdot \Delta \dot{x} \right) dt = 0.$
- Integrating the second term by parts, we conclude that

$$\int \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) \right) \cdot \Delta x\, dt = 0.$$

- This must be true for $\Delta x(t) \approx \delta(t - t_0)$, so

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = 0.$$

- The resulting equations are known as *Euler-Lagrange equations.*

32. Variational Equations (cont-d)

- *Reminder:* $\dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = 0.$

- In the Newton's case, $L = V(x) - \dfrac{1}{2} \cdot m \cdot \displaystyle\sum_{i=1}^{3} \left(\dfrac{dx_i}{dt}\right)^2.$

- Here, $\dfrac{\partial L}{\partial x_i} = \dfrac{\partial V}{\partial x_i}$, $\dfrac{\partial L}{\partial \dot{x}_i} = -m \cdot \dfrac{dx_i}{dt}$, so Euler-Lagrange's equations take the form $\dfrac{\partial V}{\partial x} + m \cdot \dfrac{d}{dt}\left(\dfrac{dx_i}{dt}\right) = 0.$

- This is equiv. to Newton's equations $m \cdot \dfrac{d^2 x_i}{dt^2} = -\dfrac{\partial V}{\partial x_i}.$

- In the general case, Euler-Lagrange equations take the form $\dfrac{\partial L}{\partial \varphi} - \displaystyle\sum_{i=1}^{3} \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial L}{\partial \varphi_{,i}}\right) = 0$, where $\varphi_{,i} \stackrel{\text{def}}{=} \dfrac{\partial \varphi}{\partial x_i}.$