Which Point From an Interval Should We Choose?

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1. Formulation of the Practical Problem

- One of the main objectives of science and engineering is to provide an optimal decision in different situations.
- In practice, we often have an algorithm that:
  - provides an optimal decision based
  - under the condition that we know the exact values of the corresponding parameters x.
- In practice, we usually know x with some uncertainty.
- For example, often, we only know an interval $[\underline{x}, \overline{x}]$ that contains the actual (unknown) value x.
- In the case of interval uncertainty, we can implement decisions corresponding to different values $x \in [\underline{x}, \overline{x}]$.
- Which value should we choose?
- Often, practitioners select the midpoint, but is this selection the best choice?

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2. Formulation of the Problem in Precise Terms

- In general, we need to make a decision u based on the state x of the system.
- According to decision theory:
  - a rational person selects a decision
  - that maximizes the value of an appropriate function known as *utility*.
- We will consider situations when:
  - for each state x and for each decision u,
  - we know the value of the utility $f(x, u)$ corresponding to us choosing u.
- Then, an optimal decision $u_{\text{opt}}(x)$ corresponding to x is the decision for which this utility is the largest:

$$f(x, u_{\text{opt}}(x)) = \max_u f(x, u).$$

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3. Formulation of the Problem (cont-d)

- In practice, we rarely know the exact state of the system, we usually know this state with some uncertainty.
- Often, we do not know the probabilities of different possible states x, we only know the bounds on x.
- In this paper, we will consider the simplest case:
  - when the state is characterized by a single parameter, i.e., when x is a real number, and
  - when a decision is described by a single number u.
- In this case, we only know that the actual (unknown) state x belongs to the interval $[\underline{x}, \overline{x}]$.
- The question is: what decision u should we make in this case?

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4. Formulation of the Problem (final)

- We also assume that the uncertainty with which we know x is relatively small.
- So in the corresponding Taylor series, we can only the first few terms in terms of this uncertainty.
- Since we already know how to compute the optimal value $u_{\text{opt}}(x)$ corresponding to x, it may be easier:
  - instead of coming up with a new algorithm that describes u as a function of the bounds $\underline{x}$ and $\overline{x}$,
  - to come up with a value s for which $u = u_{\text{opt}}(s)$.

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5. Decision Making under Interval Uncertainty

- We know the state x with uncertainty.
- Thus, for each possible decision u, we do not know the exact value of the utility.
- We only know that this utility is equal to $f(x, u)$ for some $x \in [\underline{x}, \overline{x}]$.
- Thus, all we know is that this utility value belongs to the interval

$$\left[f^-(u), f^+(u)\right] = \left[\min_{x \in [\underline{x}, \overline{x}]} f(x, u), \max_{x \in [\underline{x}, \overline{x}]} f(x, u)\right].$$

- According to decision theory, we should select u for which the following combination is the largest:

$$\alpha \cdot f^+(u) + (1 - \alpha) \cdot f^-(u).$$

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6. Decision Making under Interval Uncertainty (cont-d)

- Here, $\alpha = 1$ means that the decision maker is a complete optimist.
- In other words, he/she only takes into account the best-case situations.
- $\alpha = 0$ means that the decision maker is a complete pessimist.
- In other words, he/she only takes into account the worst-case situations.
- $\alpha \in (0, 1)$ means that the decision maker takes into account both worst-case and best-case scenarios.

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7. Resulting Formulation of the Problem.

- In these terms, our goal is:
  - given the function $f(x, u)$ and the bounds $\underline{x}$ and $\overline{x}$,
  - to find the value u for which the following objective function takes the largest possible value:

$$\alpha \cdot \max_{x \in [\underline{x}, \overline{x}]} f(x, u) + (1 - \alpha) \cdot \min_{x \in [\underline{x}, \overline{x}]} f(x, u) \to \max_{u} .$$

- Alternatively, we need to find s for which $u = u_{\text{opt}}(s)$ maximizes the above objective function.

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8. Description of the Solution

- The solution depends on whether $f(x, u)$ is increasing or decreasing with respect to x.
- If the objective function is an increasing function of x, then we should select a solution corresponding to

$$x = \alpha \cdot \overline{x} + (1 - \alpha) \cdot \underline{x}.$$

- If the objective function is a decreasing function of x, then we should select a solution corresponding to

$$x = \alpha \cdot \underline{x} + (1 - \alpha) \cdot \overline{x}.$$

- Thus, the usual selection of the midpoint s is only optimal for decision makers for which $\alpha = 0.5$.
- Intuitively, the above solution is in good accordance with the Hurwicz criterion.
- This is, however, not a proof: Hurwicz formula combines utilities, not parameter values.

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9. Future Work

- We have provided a solution for the simplest case, when we have:
  - only one parameter x describing the system and
  - only one parameter u describing possible alternatives.
- It's desirable to extend our solution to the case when we have several parameters x and several parameters u.
- What if we also have partial information about the probabilities of different values x from the interval?
- What if we have fuzzy information about x?
- In terms of α-cuts, what if we have different intervals $[\underline{x}, \overline{x}]$ for different levels of certainty α.

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11. Proof

- Let us denote the midpoint $\dfrac{\underline{x}+\overline{x}}{2}$ of the interval $[\underline{x}, \overline{x}]$ by x_0.
- Then, each point x from this interval can be represented as $x = x_0 + \Delta x$, where we denoted $\Delta x \stackrel{\text{def}}{=} x - x_0$.
- The range of possible values of Δx is $[\underline{x} - x_0, \overline{x} - x_0] = [-\Delta, \Delta]$, where we denoted $\Delta \stackrel{\text{def}}{=} \dfrac{\overline{x} - \underline{x}}{2}$.
- The difference Δx is small, so we should be able to keep only the few first terms in Δx.
- When x is known exactly, the optimal decision is $u_{\text{opt}}(x)$.
- Uncertainty is assumed to be small.
- Thus, the optimal decision u should be close to the optimal decision $u_0 \stackrel{\text{def}}{=} u_{\text{opt}}(x_0)$ corresponding to x_0.

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12. Proof (cont-d)

- So, the difference $\Delta u \stackrel{\text{def}}{=} u - u_0$ should also be small.
- In terms of Δu, the original value u has the form $u = u_0 + \Delta u$.
- Substituting $x = x_0 + \Delta x$ and $u = u_0 + \Delta u$ into $f(x, u)$ and keeping quadratic terms, we get:

$$f(x, u) = f(x_0 + \Delta x, u_0 + \Delta u) =$$

$$f(x_0, u_0) + f_x \cdot \Delta x + f_u \cdot \Delta u +$$

$$\frac{1}{2} \cdot f_{xx} \cdot (\Delta x)^2 + f_{xu} \cdot \Delta x \cdot \Delta u + \frac{1}{2} \cdot f_{uu} \cdot (\Delta u)^2,$$

$$\text{where } f_x \stackrel{\text{def}}{=} \frac{\partial f}{\partial x}(x_0, u_0), \quad f_u \stackrel{\text{def}}{=} \frac{\partial f}{\partial u}(x_0, u_0),$$

$$f_{xx} \stackrel{\text{def}}{=} \frac{\partial^2 f}{\partial x^2}(x_0, u_0), \quad f_{xu} \stackrel{\text{def}}{=} \frac{\partial^2 f}{\partial x \partial u}(x_0, u_0),$$

$$f_{uu} \stackrel{\text{def}}{=} \frac{\partial^2 f}{\partial u^2}(x_0, u_0).$$

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13. Proof (cont-d)

- To find an explicit expression for the objective function, we need to find:
  - the maximum and the minimum of the utility function $f(x, u)$
  - when $x \in [\underline{x}, \overline{x}]$, i.e., when $\Delta x \in [-\Delta, \Delta]$.
- To find the maximum and the minimum of a function of an interval, it is useful to compute its derivative.
- For our utility function we have

$$\frac{\partial f}{\partial x} = f_x + f_{xx} \cdot \Delta x + f_{xu} \cdot \Delta u.$$

- In general, the value f_x is different from 0; we will ignore a possible degenerate case when $f_x = 0$.
- Since Δx and Δu are small, their linear combination is smaller than $|f_x|$.

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14. Proof (cont-d)

- Thus, on the whole interval $\Delta x \in [-\Delta, \Delta]$, the sign of the derivative $\dfrac{\partial f}{\partial x}$ is the same as $s_x \stackrel{\text{def}}{=} \text{sign}(f_x)$.
- When $f_x > 0$ and $s_x = +1$, then:
  - the function $f(x, u)$ is an increasing function of x;
  - its maximum is attained when x is attained its largest possible values $\overline{x}$, i.e., when $\Delta x = \Delta$, and
  - its minimum is attained when $\Delta x = -\Delta$.
- When $f_x < 0$ and $s_x = -1$, then:
  - the function $f(x, u)$ is an decreasing function of x;
  - its maximum is attained when x is attained its smallest possible values $\underline{x}$, i.e., when $\Delta x = -\Delta$,
  - and its minimum is attained when $\Delta x = \Delta$.

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- In both cases, the maximum of $f(x,u)$ is attained when $\Delta x = s_x \cdot \Delta$ and its minimum when $\Delta x = -s_x \cdot \Delta$:

$$\max_{x \in [\underline{x}, \overline{x}]} f(x,u) = f(x_0 + s_x \cdot \Delta, u_0 + \Delta u) =$$

$$f(x_0, u_0) + f_x \cdot s_x \cdot \Delta + f_u \cdot \Delta u +$$

$$\frac{1}{2} \cdot f_{xx} \cdot (\Delta)^2 + f_{xu} \cdot s_x \cdot \Delta \cdot \Delta u + \frac{1}{2} \cdot f_{uu} \cdot (\Delta u)^2;$$

$$\min_{x \in [\underline{x}, \overline{x}]} f(x,u) = f(x_0 - s_x \cdot \Delta, u_0 + \Delta u) =$$

$$f(x_0, u_0) - f_x \cdot s_x \cdot \Delta + f_u \cdot \Delta u +$$

$$\frac{1}{2} \cdot f_{xx} \cdot (\Delta)^2 - f_{xu} \cdot s_x \cdot \Delta \cdot \Delta u + \frac{1}{2} \cdot f_{uu} \cdot (\Delta u)^2.$$

- Therefore, our objective function takes the form

$$\alpha \cdot \max_{x \in [\underline{x}, \overline{x}]} f(x,u) + (1-\alpha) \cdot \min_{x \in [\underline{x}, \overline{x}]} f(x,u) =$$

$$f(x_0, u_0) + (2\alpha - 1) \cdot f_x \cdot s_x \cdot \Delta + f_u \cdot \Delta u +$$

$$\frac{1}{2} \cdot f_{xx} \cdot (\Delta)^2 + (2\alpha - 1) \cdot f_{xu} \cdot s_x \cdot \Delta \cdot \Delta u + \frac{1}{2} \cdot f_{uu} \cdot (\Delta u)^2.$$

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15. Proof (cont-d)

- Let us differentiate this expression with respect to u and equate the derivative to 0:

$$f_u + (2\alpha - 1) \cdot f_{xu} \cdot s_x \cdot \Delta + f_{uu} \cdot \Delta u_{\max} = 0.$$

- Thus, $\Delta u_{\max} = -\dfrac{f_u + (2\alpha - 1) \cdot f_{xu} \cdot s_x \cdot \Delta}{f_{uu}}$.
- Let us now take into account that for each x, $f(x, u)$ attains its maximum at the known value $u_{\text{opt}}(x)$.
- Differentiating $f(x, u)$ with respect to u and equating the derivative to 0, we get:

$$f_u + f_{xu} \cdot \Delta x + f_{uu} \cdot \Delta u = 0.$$

- For $x = x_0$, i.e., when $\Delta x = 0$, this maximum is attained when $u = u_0$, i.e., when $\Delta u = 0$.

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16. Proof (cont-d)

- Substituting $\Delta x = 0$ and $\Delta u = 0$ into the above formula, we conclude that $f_u = 0$, and thus,

$$\Delta u_{\max} = -\frac{(2\alpha - 1) \cdot f_{xu} \cdot s_x \cdot \Delta}{f_{uu}}.$$

- In general, we can similarly expand $u_{\text{opt}}(x)$ in Taylor series and keep only a few first terms in this expansion:

$$u_{\text{opt}}(x) = u_{\text{opt}}(x_0 + \Delta x) = u_0 + u_x \cdot \Delta x, \text{ where } u_x \stackrel{\text{def}}{=} \frac{\partial u_{\text{opt}}}{\partial x}.$$

- Thus, for the optimal decision, $\Delta u = u_{\text{opt}}(x) - u_0 = u_x \cdot \Delta x$.
- Substituting this expression and $f_u = 0$ into the above formula, we get $f_{xu} \cdot \Delta x + f_{uu} \cdot u_x \cdot \Delta x = 0$ for all Δx.
- Thus, $f_{xu} + f_{uu} \cdot u_x = 0$, and $\dfrac{f_{xu}}{f_{uu}} = -u_x$.

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17. Proof (cont-d)

- Substituting this expression for u_x into the formula for $\Delta u_{\max}$, we conclude that:

$$\Delta u_{\max} = (2\alpha - 1) \cdot u_x \cdot s_x \cdot \Delta.$$

- Let us describe this solution in terms of the value $s \in [\underline{x}, \overline{x}]$ for which $u(s) = u_0 + \Delta u_{\max}$.
- Since $s \in [\underline{x}, \overline{x}]$, we can represent s as $s = x_0 + \Delta s$, where $\Delta s \stackrel{\text{def}}{=} s - x_0$.
- Thus, $u(s) = u(x_0 + \Delta s) = u_0 + u_x \cdot \Delta s$.
- Equating $u(s)$ and $u_0 + \Delta u_{\max}$ and using the above formula for $\Delta u_{\max}$, we conclude that

$$u_x \cdot \Delta s = (2\alpha - 1) \cdot u_x \cdot s_x \cdot \Delta.$$

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18. Proof (final)

- We get: $u_x \cdot \Delta s = (2\alpha - 1) \cdot u_x \cdot s_x \cdot \Delta$.
- Thus, $\Delta s = (2\alpha - 1) \cdot s_x \cdot \Delta$, where $\Delta = \dfrac{\overline{x} - \underline{x}}{2}$.
- Here, $s = x_0 + \Delta s = \dfrac{\underline{x} + \overline{x}}{2} + \Delta s$.
- If $f_x > 0$ and $s_x = +1$, then

$$s = x_0 + \Delta s = \frac{\underline{x} + \overline{x}}{2} + (2\alpha - 1) \cdot \frac{\overline{x} - \underline{x}}{2} =$$

$$\alpha \cdot \overline{x} + (1 - \alpha) \cdot \underline{x}.$$

- If $f_x < 0$, then we similarly get

$$s = \alpha \cdot \underline{x} + (1 - \alpha) \cdot \overline{x}. \text{ Q.E.D.}$$

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