Can We Detect Crisp Sets Based Only on the Subsethood Ordering of Fuzzy Sets? Fuzzy Sets And/Or Crisp Sets Based on Subsethood of Interval-Valued Fuzzy Sets?

Christian Servin[1], Gerardo Muela[2], and Vladik Kreinovich[2]

[1]Computer Science and Information Technology Systems Department
El Paso Community College, 919 Hunter, El Paso, TX 79915, USA
cservin@gmail.com

[2]Department of Computer Science, University of Texas at El Paso
El Paso, TX 79968, USA. gdmuela@miners.utep.edu, vladik@utep.edu

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

1. Formulation of the Problem

- A *fuzzy set* is a function $\mu : U \to [0,1]$ from some set U (*Universe of discourse*) to the interval $[0,1]$.
- This function is also known as a *membership function*.
- A fuzzy set A with a memb. f-n $\mu_A(x)$ is a *subset* of a fuzzy set B if $\mu_A(x) \le \mu_B(x)$ for all x.
- $\subseteq$ is an *order* in the sense that it is:
  - reflexive ($A \subseteq A$),
  - asymmetric ($A \subseteq B$ and $B \subseteq A$ imply $A = B$),
  - transitive ($A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$).
- Traditional (*crisp*) sets S are particular cases of fuzzy sets: $\mu_S(x) = 1$ if $x \in S$ and $\mu_S(x) = 0$ if $x \notin S$.
- A natural question: can we detect crisp sets based only on the subsethood ordering of fuzzy sets?

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

2. Formulation of the Problem in Precise Terms

- Suppose now that we have a class F of all fuzzy sets with the subsethood ordering $A \subseteq B$.
- However, we have no access to the actual values of the corresponding membership functions.
- Based only on this ordering relation $A \subseteq B$, can we then detect crisp sets?
- In this talk, we show that this is indeed possible.
- We also show that based on $\subseteq$ for interval-valued fuzzy sets, we can:
  - detect crisp sets, and
  - type-1 fuzzy sets.

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

3. Let Us First Consider $[0, 1]$-Based Fuzzy Sets

- First, we will prove that the empty set $\emptyset$ can be uniquely determined based on the subsethood relation.
- Second, we will show that 1-element crisp sets, i.e., sets of the type $\{x_0\}$, can be thus determined.
- Then, we consider 1-element fuzzy sets, for which for some $x_0 \in U$, $\mu_A(x_0) > 0$ and $\mu_A(x) = 0$ for all $x \neq x_0$.
- Then, we will prove that 1-element fuzzy sets can be determined based on the subsethood relation.
- Finally, we prove that crisp sets can be uniquely determined based on the subsethood relation.

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

4. How to Detect the Empty Set and 1-Element Crisp Sets

- An empty set $\emptyset$ is a fuzzy set for which $\mu_\emptyset(x) = 0$ for all $x \in U$.
- A fuzzy set A is an empty set if and only if $A \subseteq B$ for all fuzzy sets B.
- A non-empty fuzzy set A is a one-element crisp set if and only if the following two conditions are satisfied:
  - the class $\{B : B \subseteq A\}$ is linearly ordered and
  - for no proper superset A' of A, the class $\{B : B \subseteq A'\}$ is linearly ordered.
- Indeed, for $A = \{x_0\}$, all subsets B are fuzzy 1-element sets, so $\{B : B \subseteq A'\}$ is linearly ordered.

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

5. How to Detect 1-Element Crisp Sets (cont-d)

- If $\mu_A(x_1) > 0$ and $\mu_A(x_2) > 0$, then we can take:
  - $\mu_{B_1}(x_1) = \mu_A(x_1)$ and $\mu_{B_1}(x) = 0$ for all other x, and
  - $\mu_{B_2}(x_2) = \mu_A(x_2)$ and $\mu_{B_2}(x) = 0$ for all other x.
- Then, $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$.
- So, $\mu_A(x) < 0$ for only one x.
- If $\mu_A(x) < 1$, then $A \subset A' = \{x\}$ and $\{B : B \subseteq A'\}$ is linearly ordered.
- Thus, $\mu_A(x) = 1$, so A is a 1-element crisp set.
- So, we can indeed detect 1-element crisp sets based on the subsethood relation $\subseteq$.

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

6. How to Detect 1-Element Fuzzy Sets and Crisp Sets

- A 1-element fuzzy set A can be detected as a non-empty subset of a 1-element crisp set.
- Crisp sets can be detected as follows:

$$A \text{ is crisp} \Leftrightarrow \forall A\,(B \text{ is a one-element fuzzy subset of } A \Rightarrow$$

$$\exists C\,((B \subseteq C \subseteq A)\,\&\,(C \text{ is a 1-element crisp set}))).$$

$\Rightarrow$ This is clearly true for crisp sets.

$\Leftarrow$ Vice versa, if $0 < \mu_A(x_0) < 1$ for some x_0, then:

- the set $B \stackrel{\text{def}}{=} A \cap \{x_0\}$ is a 1-element fuzzy subset of A, and
- the set B cannot be embedded in a 1-element crisp subset of A.

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

7. What About Interval-Valued Fuzzy Sets?

- An empty set $\emptyset$ is an interval-valued fuzzy set for which $\mu_\emptyset(x) = [0, 0]$ for all $x \in U$.
- An interval-valued fuzzy set A is an empty set if and only if $A \subseteq B$ for all interval-valued fuzzy sets B.
- We say that A is *special* if for some x_0 and $a > 0$, $\mu_A(x_0) = [0, a]$ and $\mu_A(x) = 0$ for all $x \neq x_0$.
- A non-empty interval-valued fuzzy set A is special if and only if the class $\{B : B \subseteq A\}$ is linearly ordered.
- Indeed, if $\mu_A(x_1) \neq [0, 0]$ and $\mu_A(x_2) \neq [0, 1]$, then for $B_1 = A \cap \{x_1\}$ and $B_2 = A \cap \{x_2\}$, we have

$$B_1 \not\subset B_2 \text{ and } B_2 \not\subset B_1.$$

- If $\mu_A(x) = [\underline{a}, \overline{a}]$ with $\underline{a} > 0$, then for $B_1 = [0, \overline{a}]$ and $B_2 = [0.5 \cdot \overline{a}, 0.5 \cdot \overline{a}]$, we have $B_1 \not\subset B_2$ and $B_2 \not\subset B_1$.

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

8. Interval-Valued Fuzzy Sets (cont-d)

- We say that A is a *1-element type-1* fuzzy set if for some x_0 and $a > 0$:

$$\mu_A(x_0) = [a, a] \text{ and } \mu_A(x) = [0, 0] \text{ for all } x \neq x_0.$$

- A non-empty A is a 1-element type-1 set if and only if it is satisfies the following three properties:
  - the set A is not special (in the sense of the above definition),
  - there exists a special set $B \subseteq A$ for which the class $\{C : B \subseteq C \subseteq A\}$ is linearly ordered, and
  - for no proper superset A' of A, the class $\{C : B \subseteq C \subseteq A'\}$ is linearly ordered.

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

9. Interval-Valued Fuzzy Sets: Final Result

- Since we have subsethood, we also have union: the union of A_β is the $\subseteq$-smallest set that contains all A_β.
- We can thus define type-1 fuzzy sets as unions of 1-element type-1 fuzzy sets.
- Once we can detect type-1 fuzzy sets, we can use techniques from the previous sections to detect crisp sets.
- Thus:
  - *we can indeed detect type-1 fuzzy sets and crisp sets*
  - *based only on subsethood relation between interval-valued fuzzy sets.*

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

10. Open Question

- A similar question can be asked if we only consider normalized fuzzy sets.
- To be more precise:
  - suppose that we have a class $\mathcal{N}$ of all normalized fuzzy sets, and
  - we have a subsethood ordering relation $A \subseteq B$ between these sets – but
  - we do not have access to the actual values of the corresponding membership functions.
  - Based only on this ordering relation $A \subseteq B$, can we then detect crisp sets?

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question

11. Acknowledgments

This work was supported in part by grant HRD-1242122 from the US National Science Foundation.

Formulation of the . . .
Formulation of the . . .
Let Us First Consider . . .
How to Detect the . . .
How to Detect 1- . . .
How to Detect 1- . . .
What About Interval- . . .
Interval-Valued Fuzzy . . .
Open Question