Towards Foundations of Fuzzy Utility: Taking Fuzziness into Account Naturally Leads to Intuitionistic Fuzzy Degrees

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1. Need to Help People Make Decisions

- In many practical situations, we need to make a decision.
- In other words, we need to select an alternative which is, for us, better than all other possible alternatives.
- If the set of alternatives is small, we can easily make such a decision: indeed,
  - we can easily compare each alternative with every other one, and,
  - based on these comparisons, decide which one is better.
- However, when the number of alternatives becomes large, we have trouble making decisions.

2. Need to Help People Make Decisions (cont-d)

- Even in simple situations, when we are looking for cereal in a supermarket:
  - there are usually so many selections
  - that we just ignore most of them and go with a familiar one – instead of the optimal one.
- The situation is even more complicated if:
  - we are trying to make a decision not on behalf of ourselves,
  - but rather on behalf of a company or a community.

3. Need to Help People Make Decisions (cont-d)

- In this case, even comparing two alternatives is not easy:
  - it requires taking into account interests of different people involved,
  - so the decision making process becomes even more complicated.

4. Traditional Approach to Decision Making: the Notion of Utility

- The traditional approach to decision making was originally motivated by the idea of money.
- When money was invented, it was a revolutionary idea that made economic exchange much easier.
- Before money was invented, people exchanged goods by barter: chicken for a shirt, jewelry for boots, etc.
- Thus, to make a proper decision, every person needed to be able to compare every two items with each other:
  - how many chickens is this person willing to exchange for a shirt,
  - how many boots for a golden earing, etc.
- For n goods, we have $\dfrac{n \cdot (n-1)}{2} \approx \dfrac{n^2}{2}$ possible pairs.

5. The Notion of Utility (cont-d)

- So, each person had to have in mind a table of $n^2/2$ numbers.
- With money as a universally accepted means of exchange, all the person needs to do is to decide:
  - for each of n items,
  - how much he or she is willing to pay for 1 unit.
- So, to successfully make decisions, it is sufficient to know n numbers – the values of each of n items; then:
  - even when we want to barter,
  - we can easily decide how many chickens are worth a shirt:
  - it is sufficient to divide the price of a shirt by the price of a chicken.

6. The Notion of Utility (cont-d)

- A similar idea can be used to compare different alternatives.
- All we need is to have a numerical scale, i.e., we need a 1-parametric family of "standard" alternatives
  - whose quality increases
  - with the increase in the value of the parameter.
- This can be the money amount.
- Alternatively, this can be the probability p of a lottery in which we get something very good.
- The larger the probability, the more preferable the lottery.

7. The Notion of Utility (cont-d)

- Then, instead of comparing every alternative a with every other alternative, we simply compare:
  - every alternative with
  - alternatives on the selected scale.
- Thus, for each a, we find the numerical value of the standard alternative which is equivalent to a.
- This numerical value is known as the *utility* $u(a)$ of a given alternative a.

8. The Notion of Utility (cont-d)

- In terms of utility, an alternative a is better than the alternative a' if and only if $u(a) > u(a')$; thus:
  - once we have found the utility $u(a)$ of each alternative,
  - then it is easy to predict which alternative the person will select:
  - he/she will select the alternative for which the utility $u(a)$ is the largest possible.

9. How to Actually Find the Utility

- The fastest way to find the utility of a given alternative a based on binary comparisons is to use bisection.
- Usually, we have an a prior lower bound and an a priori upper bound for the desired utility $u(a)$: $\underline{u} \le u(a) \le \overline{u}$.
- In other words, we know that the desired utility $u(a)$ is somewhere in the interval $[\underline{u}, \overline{u}]$.
- In this procedure, we will narrow down this interval.
- Once an interval is given, we can:
  - compute its midpoint $\widetilde{u} = \dfrac{\underline{u} + \overline{u}}{2}$ and
  - compare a with the corresponding standard alternative $s(\widetilde{u})$.
- If a is exactly equivalent to $s(\widetilde{u})$, then $u(a) = \widetilde{u}$.

10. How to Actually Find the Utility (cont-d)

- However, such exact equivalences are rare; in most cases, we will find out that:
  - either a is better than $s(\widetilde{u})$; we will denote it by $s(\widetilde{u}) < a$; or
  - the standard alternative is better: $a < s(\widetilde{u})$.
- In the first case, the preference $s(\widetilde{u}) < a$ means that

$$\widetilde{u} < u(a).$$

- Thus, we know that $u(a) \in [\widetilde{u}, \overline{u}]$.
- In other words, we have a new interval containing the desired utility.
- We can obtain this new interval if we replace the previous lower bound $\underline{u}$ with the new lower bound $\widetilde{u}$.

11. How to Actually Find the Utility (cont-d)

- In the second case, the preference $a < s(\widetilde{u})$ means that

$$u(a) < \widetilde{u}.$$

- Thus, we know that $u(a) \in [\underline{u}, \widetilde{u}]$.
- In other words, we have a new interval containing the desired utility.
- We can obtain this new interval if we replace the previous upper bound $\overline{u}$ with the new upper bound $\widetilde{u}$.
- In both cases, the width of the interval is decreased by a factor of 2.
- Then, we can repeat this procedure, and in k steps, we get $u(a)$ with accuracy 2^{-k}.
- For example, in 7 steps, we get an accuracy of 1%.

12. Need to Take Fuzziness into Account

- The above procedure works well if a person is absolutely sure about his/her preferences.
- In practice, we are often not 100% sure about our preferences.
- This is especially true when we compare alternatives of similar value.
- It is reasonable to describe this uncertainty in fuzzy terms.
- For example, if we use money as a standard scale, then for each alternative a,
  - instead of having a single amounts of money equivalent to this item,
  - we may have different amounts with different degree of certainty.

13. Need to Take Fuzziness into Account (cont-d)

- In other words,
  - instead of the above crisp model, in which a person has an exact utility value $u(a)$ for each a,
  - for each person and for each alternative a, we have a membership function $\mu_a(u)$,
  - this function describes, for each possible value u, to what extend $s(u)$ is equivalent to a.

14. How to Elicit Fuzzy Utility: a Reasonable Idea

- We know how to elicit crisp utility $u(a)$ of a given alternative a: we need to compare:
  - the alternative a
  - with different values u_0 of the scale.
- In the case of fuzzy utility, it is reasonable to apply the same procedure.
- The only difference is that:
  - now, since the utility value $u(a)$ is fuzzy,
  - this comparison will not lead to a crisp "yes"-"no" answer;
  - instead, we will get a fuzzy answer – the degree to which it is possible that a is better than u_0,
  - and, if needed, the degree to which it is possible that a is worse than u_0.

15. Remaining Open Problems

- In the crisp case, we can determine the utility value $u(a)$ from the results of the user's comparisons.
- To deal with the more realistic fuzzy case, we need:
  - to extract the fuzzy utility
  - from the fuzzy answers to different comparisons.
- This is the question that we deal with in this talk.
- Interestingly, it turns out that in this context, intuitionistic fuzzy degrees naturally appear.
- In other words, instead of a single degree of confidence in each statement, we now get *two* degrees:
  - the degree to which this statement is true, and
  - the degree to which this statement is false.
- These degrees do not add up to 1.

16. What If We Compare the Alternative a with a Fixed Value u_0 on the Utility Scale?

- In the crisp case, each alternative a is equivalent to a single number $u(a)$ on the utility scale.
- In general, the utility of an alternative is characterized:
  - not by a single number,
  - but rather with a membership function $\mu_a(u)$.
- This function describes, for each value u from the utility scale, to what extent a is equivalent to u.
- What will happen is we compare the alternative a to a value u_0 on the utility scale?
- In the crisp case, the changes that a is exactly equivalent to a_0 are slim.
- So, we have either $a < u_0$ or $u_0 < a$.

17. What If We Compare a with u_0 (cont-d)

- Thus, we can ask whether a is better than u_0, or we can ask whether u_0 is better than a.
- Whatever question we ask, we get the exact same information.
- Let us first consider the question of whether a is better than u_0, i.e., whether $u_0 < a$.
- How can we extend this to the fuzzy case?
- It is convenient to take into account that:
  - while from math. viewpoint, $<$ is a relation,
  - and in mathematics, relations usually treated differently than functions,
  - from the computational viewpoint, $<$ a function.

18. What If We Compare a with u_0 (cont-d)

- Similarly:
  - the addition $+$ is a function that takes two numbers and returns a number which is their sum,
  - the relation $<$ is a function that takes two numbers and returns a boolean value: true or false.
- Since $<$ can be naturally treated as function:
  - the question of how to extend this to fuzzy
  - becomes a particular case of a more general question:
  - how to extend functions to fuzzy?
- This extension is well known, it is described by Zadeh's extension principle.
- Let us recall how this principle is usually derived.

19. Zadeh's Extension Principle and How It Is Usually Derived

- Suppose that:
  - we have a function $y = f(x_1, \ldots, x_n)$ of n real-valued variables, and
  - we have fuzzy information about the values $x_1, \ldots, x_n$,
  - i.e., we know membership functions $\mu_1(x_1), \ldots, \mu_n(x_n)$ that describes our knowledge about the inputs $x_1, \ldots, x_n$.
- Based on this information, what do we know about $y = f(x_1, \ldots, x_n)$?
- Intuitively, Y is a possible value of the variable y if there exists values $X_1, \ldots, X_n$ for which:
  - the value X_1 is a possible value of x_1 *and* ...
  - *and* X_n is a possible value of x_n
  - *and* $Y = f(X_1, \ldots, X_n)$.

20. Zadeh's Extension Principle (cont-d)

- We know the degrees $\mu_i(X_i)$ to which each each real number X_i is a possible values of the input x_i.
- We need to combine these degrees into our degree of confidence in a composite and-statement.
- For this, we can use an "and"-operation (t-norm).
- The simplest of them is $\min(a, b)$.
- Thus, for each $(X_1, \ldots, x_n)$ for which $Y = f(X_1, \ldots, X_n)$, our degree of confidence is the above and-statement is

$$\min(\mu_1(X_1), \ldots, \mu_n(X_n)).$$

- The existential quantifier "there exists" is, in effect, an "or".
- It means that either this property is true for one tuple, or for another tuple, etc.

21. Zadeh's Extension Principle (cont-d)

- Thus:
  - to find the degree to which the value Y is possible,
  - we need to apply an "or"-operation (t-conorm)
  - to the degrees of confidence of the corresponding "and"-statements.
- The simplest "or"-operation is $\max(a,b)$.
- Thus, we arrive at the following formula for the degree $\mu(Y)$ to which Y is a possible value of the variable y:

$$\mu(Y) = \max\{\min(\mu_1(X_1), \ldots, \mu_n(X_n)) : f(X_1, \ldots, X_n) = Y\}.$$

- This formula – first proposed by L. Zadeh himself – is known as *Zadeh's extension principle.*

22. Let Us Apply Zadeh's Extension Principle to Our Problem: Resulting Formulas

- In our case, we have a Boolean-valued function $f(x_1, x_2) = (x_1 < x_2)$ of $n = 2$ real-valued variables.
- We compare an alternative a with fuzzy utility $\mu_a(u)$ with a crisp value u_0.
- Zadeh's extension principle takes the following form:
  - for the value $y =$"true", the degree $\mu_+(a < u_0)$ that the statement $a < u_0$ is true is equal to

$$\mu_+(a < u_0) = \max(\mu_a(u) : u < u_0);$$

  - for the value $y =$"false", the degree $\mu_-(a < u_0)$ that the statement $a < u_0$ is false is equal to

$$\mu_-(a < u_0) = \max(\mu_a(u) : u \geq u_0).$$

23. Let Us Analyze the Resulting Formulas

- In fuzzy logic negation is represented by $1 - a$.
- Thus, our degree of believe that A is false is estimated as 1 minus degree that A is true.
- So, we should expect that $\mu_+(a < u_0) + \mu_-(a < u_0) = 1$.
- Let us show, however, that this is not the case.
- Indeed, let us consider a typical case when $\mu_a(u)$ is a fuzzy number, i.e., when for some value U:
  - the function $\mu_a(u)$ increases to 1 when $u \le U$, and
  - this function decreases from 1 when $u \ge U$.
- When $u_0 < U$, then the function $\mu_a(u)$ is increasing for all $u < u_0$ and thus, $\mu_+(a < u_0) = \mu_a(u_0)$.
- On the other hand, since $u_0 < U$ and for $u = U$, we have $\mu_a(U) = 1$, we get $\mu_-(a < u_0) = 1$; thus:

$$\mu_+(a < u_0) + \mu_-(a < u_0) = 1 + \mu_a(u) \neq 1.$$

24. Analyzing the Resulting Formulas (cont-d)

- The only exception is, of course, we consider absolutely impossible values u for which $\mu_a(u) = 0$.
- Similarly, when $u_0 \geq U$, then the function $\mu_a(u)$ is decreasing for all $u > u_0$ and thus,

$$\mu_-(a < u_0) = \mu_a(u_0).$$

- On the other hand, since $u_0 \geq U$ and for $u = U$, we have $\mu_a(U) = 1$, we get $\mu_+(a < u_0) = 1$.
- Thus, in this case too, we have

$$\mu_+(a < u_0) + \mu_-(a < u_0) = 1 + \mu_a(u) \neq 1.$$

- The only exception is absolutely impossible values u for which $\mu_a(u) = 0$.

25. So, We Get Intuitionistic Fuzzy Degrees

- In the traditional fuzzy logic:
  - the sum of degrees to which each statement is true and to which this same statement is false
  - is always equal to 1.
- This means that when we compare alternatives, we get beyond the traditional fuzzy logic.
- How can we describe where we are?
- This is not the only case when the degrees of confidence in a statement and in its negation do not add up to 1.
- To describe such cases, K. Atanassov came up with an idea of *intuitionistic fuzzy logic*.

26. Intuitionistic Fuzzy Degrees (cont-d)

- In this logic, for each statement, we have *two* degrees:
  - the degree to which this statement is true, and
  - the degree to which this statement is false.
- These degrees do not necessarily add to 1.
- So, the result of comparing two alternatives is an intuitionistic fuzzy degree.

27. What We Got Is Somewhat Different From Intuitionistic Fuzzy Logic

- There is a minor difference between what we observe and the traditional intuitionistic fuzzy logic:
  - in the intuitionistic fuzzy logic, the sum of positive and negative degrees is always ≤ 1, while
  - in our case, the sum is always greater than or equal to 1.
- However, such (minor) generalization of intuitionistic fuzzy logic has been proposed in the past.
- We can reconcile the results of comparing alternatives with the traditional intuitionstic fuzzy logic.
- Indeed, in general, Zadeh's extension principle, we compute the degree to which y is a *possible* value.
- In particular, $\mu_+(a < u_0)$ is the degree to which is is possible that $a < u_0$.

28. What We Got (cont-d)

- Similarly, $\mu_-(a < u_0)$ is a degree to which it is possible that $a \geq u_0$.
- Instead, we consider degrees $n_+(a < u_0)$ and $n_-(a < u_0)$ to which it is *necessary* that $a < u_0$ and that $a \geq u_0$.
- They can be defined, as usual, as 1 minus the degree to which the opposite statement is possible.
- Then, we get

$$n_+(a < u_0) = 1 - \mu_-(a < u_0) \text{ and } n_-(< u_0) = 1 - \mu_+(a < u_0).$$

- From the fact that $\mu_+(a < u_0) + \mu_-(a < u_0) \geq 1$, we can now conclude that

$$n_+(a < u_0) + n_-(a < u_0) = 2 - (\mu_+(a < u_0) + \mu_-(a < u_0)) \leq 1.$$

- Thus, the degrees of necessity are consistent with the traditional intuionistic fuzzy logic.

29. Reconstructing the Original Membership F-n from the Results of Expert Elicitation

- We assume that the expert's preferences are described by a membership function $\mu_a(u_0)$.
- As we have mentioned, as a result of expert elicitation, we do not get this function.
- Instead, we get instead a more complex construct, in which for each possible value u_0, we get two degrees:

$$\mu_+(a < u_0) \text{ and } \mu_-(a < u_0).$$

- Interestingly, from these degrees, we can uniquely reconstruct the original membership function.

30. Reconstructing the Original Membership Function (cont-d)

- Indeed, as have shown:
  - when $u_0 \leq U$, then we have $\mu_+(a < u_0) = \mu_a(u_0)$ and $\mu_-(a < u_0) = 1$; and
  - when when $u_0 \geq U$, then we have $\mu_-(a < u_0) = \mu_a(u_0)$ and $\mu_+(a < u_0) = 1$.
- In both cases, we thus have

$$\mu_a(u_0) = \min(\mu_+(a < u_0), \mu_-(a < u_0)).$$

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