Centroids Beyond Defuzzification

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1. Centroid Defuzzification: A Brief Reminder

- In fuzzy control, we start with the expert rules.
- These rules are formulated in terms in imprecise ("fuzzy") words from natural language.
- Given the inputs, we recommend what control value u to use.
- This recommendation is also fuzzy:
  - for each possible value u,
  - we provides a degree $\mu(u) \in [0, 1]$ to which u is a reasonable control.
- Such a fuzzy outcome is perfect if the main objective of the system is to advise a human controller.
- In many practical situations, however, we want this system to actually control.

2. Centroid Defuzzification (cont-d)

- In such situations, it is important to transform:
  - the fuzzy recommendation – as expressed by the function $\mu(u)$ (known as the *membership function*)
  - into a precise control value $\overline{u}$ that this system will apply.
- Such a transformation is known as *defuzzification*.
- The most widely used defuzzification procedure is *centroid defuzzification* $\overline{u} = \dfrac{\int u \cdot \mu(u)\, du}{\int \mu(u)\, du}$.

3. Geometric Meaning of Centroid Defuzzification

- The name for this defuzzification procedure comes from the fact that:
  - if we take the subgraph of the function $\mu(u)$, i.e., the 2-D set $S \stackrel{\text{def}}{=} \{(u, y) : 0 \le y \le \mu(u)\}$,
  - then the value $\overline{u}$ is actually the u-coordinate of this set's center of mass ("centroid") $(\overline{u}, \overline{y})$.
- In fuzzy technique, we only use the u-coordinate of the center of mass.
- A natural question is: is there a fuzzy-related meaning of the y-coordinate $\overline{y}$?
- In this talk, we describe such a meaning.

4. Mathematical Formula for the y-Component

- In general, the y-component of the center of mass of a 2-D body S has the form $\overline{y} = \dfrac{\int_S y\,du\,dy}{\int_S du\,dy}$.
- The denominator is the same as for the u-component: it is equal to $\int \mu(u)\,du$.
- The numerator can also be easily computed as

$$\int_S y\,du\,dy = \int_u du \cdot \int_0^{\mu(u)} y\,dy = \frac{1}{2} \cdot \int_u \mu^2(u)\,du.$$

- Thus, we have $\overline{y} = \dfrac{1}{2} \cdot \dfrac{\int \mu^2(u)\,du}{\int \mu(u)\,du}$.

5. First Meaning of This Formula

- The u-component of the centroid is the weighted average value of u, with weights proportional to $\mu(u)$.
- Similarly, the expression $\overline{y}$ is the weighted average value of $\mu(u)$.
- Each value $\mu(u)$ is the degree of fuzziness of the system's recommendation about the control value u.
- Thus, the value $\overline{y}$ can be viewed with the weighted average value of the degree of fuzziness.
- Let us show that this interpretation makes some sense.
- **Proposition.**
  - *The value $\overline{y}$ is always between 0 and 1/2.*
  - *For a measurable function $\mu(u)$, $\overline{y} = 1/2$ if and only if $\mu(u)$ is almost everywhere 0 or 1.*

6. First Meaning of This Formula (cont-d)

- In other words, if we ignore sets of measure 0:
  - the value $\overline{y}$ is equal to $1/2$ if and only if
  - the corresponding fuzzy set is actually crisp.
- For all non-crisp fuzzy sets, we have $\overline{y} < 1/2$.
- For a triangular membership function, one can check that we always have $\overline{y} = 1/3$.
- For trapezoid membership functions, $\overline{y}$ can take any possible value between $1/3$ and $1/2$.
- The larger the value-1 part, the larger $\overline{y}$.

7. Proof

- Since $\mu(u) \in [0,1]$, we always have $\mu^2(u) \le \mu(u)$, thus $\int \mu^2(u)\,du \le \int \mu(u)\,du$, hence $\dfrac{\int \mu^2(u)\,du}{\int \mu(u)\,du} \le 1.$
- So, $\overline{y} \le 1/2$.
- Vice versa, if $\overline{y} = 1/2$, then $\dfrac{\int \mu^2(u)\,du}{\int \mu(u)\,du} = 1.$
- Multiplying both sides of this equality by the denominator, we conclude that $\int \mu^2(u)\,du = \int \mu(u)\,du$, i.e.:

$$\int \left(\mu(u) - \mu^2(u)\right)\,du = 0.$$

- The difference $\mu(u) - \mu^2(u)$ is always non-negative.
- Since its integral is 0, this means that almost always $\mu(u) - \mu^2(u)$, i.e., $\mu(u) = 0$ or $\mu(u) = 1$.
- The proposition is proven.

8. A Version of the First Meaning

- In general, in the fuzzy case, we have different values of $\mu(u)$ for different u.
- We want to find a single degree μ_0 which best represents all these values.
- This is natural to interpret as requiring that:
  - the mean square difference weighted by $\mu(u)$ – i.e., the value $\int (\mu(u) - \mu_0)^2 \cdot \mu(u)\, du$
  - attains its smallest possible value.
- Differentiating the minimized expression with respect to μ_0 and equating the derivative to 0, we get:

$$\int 2(\mu_0 - \mu(u)) \cdot \mu(u)\, du = 0.$$

- Hence $\mu_0 = \dfrac{\int \mu^2(u)\, du}{\int \mu(u)\, du}$, and $y_0 = (1/2) \cdot \mu_0$.

9. Second Meaning

- Membership functions $\mu(u)$ and probability density functions $\rho(u)$ differ by their normalization:
  - for a membership function $\mu(u)$, we require that $\max\limits_u \mu(u) = 1$, while
  - for a probability density function $\rho(u)$, we require that $\int \rho(u)\,du = 1$.
- For every $f(u) \geq 0$, we can divide it by an appropriate constant c and get $\mu(u)$ or $\rho(u)$:
  - if we divide $f(u)$ by $c = \max\limits_v f(v)$, then we get a membership function $\mu(u) = \dfrac{f(u)}{\max\limits_v f(v)}$;
  - if we divide $f(u)$ by $c = \int \mu(v)\,dv$, we get a probability density function $\rho(u) = \dfrac{f(u)}{\int f(v)\,dv}$.

10. Second Meaning (cont-d)

- In particular, for each $\mu(u)$, we can construct the corresponding probability density function
$$\rho(u) = \frac{\mu(u)}{\int \mu(v)\, dv}.$$
- In terms of this expression $\rho(u)$, the formulas for both components of the center of mass are simplified.
- The result $\overline{u}$ of centroid defuzzification takes the form
$$\overline{u} = \int u \cdot \rho(u)\, du.$$
- It is simply the expected value of control under this probability distribution.
- Similarly, the value $\mu_0 = 2\overline{y}$ takes the form
$$\mu_0 = \int \mu(u) \cdot \rho(u)\, du.$$

11. Second Meaning (cont-d)

- So, μ_0 is simply the expected value of the membership function.
- Note: $\int \mu(u) \cdot \rho(u)\, du$ is Zadeh's formula for the probability of the fuzzy event.
- Reminder: $\mu(u)$ characterizes to what extent a control value u is reasonable.
- So, μ_0 is the probability that a control value selected by fuzzy control will be reasonable.

12. Second Meaning (cont-d)

- This interpretation is in good accordance with the above Proposition.
- If we are absolutely confident in our recommendations, i.e., if $\mu(u)$ is a crisp set, then:
  - the probability μ_0 is equal to 1,
  - thus, $\overline{y} = (1/2) \cdot \mu_0$ is equal to $1/2$.
- On the other hand, if we are not confident in our recommendations, then:
  - the probability μ_0 is smaller than 1 and
  - thus, its half $\overline{y}$ is smaller than $1/2$.

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