How to Reconcile Randomness with Physicists' Belief that Every Theory Is Approximate: Informal Knowledge Is Needed

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1. What Sequences of Measurement Results Can We Expect

- The usual analysis of algorithms and computability takes into account what we can compute "from scratch".
- Namely, we only us the given input and performing appropriate computational steps.
- In principle, in addition to the given input, we could also use the results of physical experiments.
- We normally do not use these results when performing unrelated computations.
- We do not expect the results of physical experiments to help us solve unrelated complex problems.

2. What Sequences Measurement Results Can We Expect (cont-d)

- It would be very strange to find, e.g.:
  - a radioactive material
  - whose emissions will solve a given NP-hard problem like propositional satisfiability.
- We may get some help in solving this problem:
  - from other people,
  - maybe from signals sent to us by an alien civilization – in short, from someone with intelligence.
- However, we do not expect such help from an (unintelligent) physical process.
- This idea was formalized by L. Levin in terms of *Kolmogorov complexity*.
- Let us briefly recall the corresponding formalization.

3. Kolmogorov Complexity: A Brief Reminder

- Computers can process many different types of objects – arrays, graphs, images, videos, etc.
- However, in a computer, every object is represented as a sequence of 0s and 1s, i.e., as a *binary sequence*.
- Kolmogorov complexity was invented to formalize another intuitive idea: of a random binary sequence.
- From the purely mathematical viewpoint:
  - if we consequently flip a coin and write down all heads as 1s and all tails as 0s,
  - then all length-n sequences of 0s and 1s have the exact same probability 2^{-n}.
- From this viewpoint, they are equally probable and thus, seem to have an equal right to be called random.

4. Kolmogorov Complexity (cont-d)

- In particular, for a large n:
  - a sequence that we will actually get after flipping the coin n times has the same probability as
  - the sequence 0101...01, in which the sequence 01 is repeated many times.
- However, intuitively:
  - the seemingly lawless sequence that we will actually get after flipping a coin looks random, while
  - the sequence 0101...01 does not look random
  - and it would be very surprising if such a sequence indeed appears as a result of flipping a coin.

5. Kolmogorov Complexity (cont-d)

- Moreover:
  - if something like this sequence will be observed in a casino,
  - after a while we will be absolutely sure that cheating is taking place, and
  - that the corresponding slot machine is not truly random (as it is supposed to be).
- Kolmogorov noticed that there is a difference between:
  - sequences which are intuitively random and
  - sequences which are not intuitively random.

6. Kolmogorov Complexity (cont-d)

- The reason why we do not believe that the sequence 0101...01 is truly random is that:
  - this sequence can be generated by a reasonably short program,
  - in which we print 01 in a loop.
- On the other hand:
  - if we consider a real sequence of 0s and 1s obtained by flipping a coin,
  - we do not expect to find any regularity there,
  - so the only way to generate this sequence is to print it bit-by-bit.
- The length of such a program is practically equal to the length $\mathrm{len}(x)$ of the binary string.

7. Kolmogorov Complexity (cont-d)

- To describe this difference in precise terms, Kolmogorov introduced the notion of Kolmogorov complexity $C(x)$.
- It is the shortest length of a program (in a fixed programming language) that generates the string x.
- In these terms, if $C(x) \ll \text{len}(x)$, this sequence is not random.
- On the other hand:
  - if $C(x) \approx \text{len}(x)$ – or, to be more precise, if $C(x) \geq \text{len}(x) - c$ for some small c,
  - then we can say that the sequence x is random.

8. Prefix Kolmogorov Complexity

- The above definition of randomness can be made even more intuitive if we take into account that:
  - we can flip the coin as many times as we want, so
  - the coin-flipping sequence can be extended to any length.
- The fact that we stopped should not change our opinion on whether this sequence is random or not.
- From this viewpoint:
  - if it is not so easy to generate the binary sequence that we have so far, but
  - much easier to generate its extension,
  - then the resulting sequence should still be marked as not random.

9. Prefix Kolmogorov Complexity (cont-d)

- Generating an extension can be easier than the sequence itself; for example:
  - to generate a sequence 001001...001001 is somewhat easier than
  - its initial segment 001001...00100 when we need to take special care of the last two 0s.
- So, we need to replace the original Kolmogorov complexity $C(x)$ with *prefix Kolmogorov complexity* $K(x)$.
- $K(x)$ is the shortest length of the program that generates either x or a sequence starting with x.

10. Conditional Kolmogorov Complexity And the Notion of Information

- We are interested in how using an auxiliary sequence y can potentially affect our computations.
- In principle:
  - when we use the bits from the given sequence y,
  - we do not need to compute them and thus,
  - it may be possible to have a shorter program for computing x.
- An analogy is that if we already have a program for computing $\sin(x)$, then computing $\sin^2(x)$ is fast.

11. Conditional Kolmogorov Complexity (cont-d)

- All you need to do is access the value of the sine and square it; in contrast:
  - if we needed to compute $\sin^2(x)$ "from scratch", by using only arithmetic operations,
  - we would need to perform a dozen or so operations – which would necessitate a much longer program.
- From this viewpoint, it is reasonable to consider program for which:
  - an access to the i-th bit from the sequence y
  - can be done by simply writing something like $y[i]$.
- The shortest length of such using-calls-to-y program is known as the *conditional prefix Kolmogorov complexity*.
- It is usually denoted by $K(x|y)$.

12. Conditional Kolmogorov Complexity (cont-d)

- In these terms:
  - the fact that the use of y should not make computing x easier means that
  - the complexity $K(x|y)$ should not be much smaller than the complexity of computing x w/o using y,
  - i.e., that we should not have $K(x|y) \ll K(x)$.
- Possibly using y does not mean that we have to; so:
  - when we compute $K(x|y)$, we consider all possible programs for computing x,
  - including programs that do not use y at all.
- Thus, $K(x|y) \leq K(x)$; so, the difference $K(x) - K(x|y)$ must be non-negative.
- This difference is known as *information* in y about x; it is denoted by $I(x : y)$.

13. Conditional Kolmogorov Complexity (cont-d)

- Similarly to the definition of randomness:
  - the fact that using y should not affect our ability to compute x
  - can be described as $I(x : y) \leq c$ for some small $c > 0$.

14. Levin's Formalization of the Above Intuitive Idea

- We want to formalize the commonsense idea that:
  - the results of physical experiments
  - should not help us solve mathematical problems.
- All mathematics can be (and have been) formulated in terms of set theory ST.
- For example, it can be formuled in terms of the usual Zermelo-Frenkel axiomatics ZF.
- In these terms:
  - solving a mathematical problem
  - i.e., checking whether a given mathematical statement is true or not
  - is equivalent to checking whether a given statement from ST is true or not.

15. Levin's Formalization (cont-d)

- Some mathematical problems are computational, not about proving results.
- Such problems can also be described in these terms.
- Namely, if we want to compute the value of a real number (such as π), this means that:
  - for each n, we must find out
  - whether the n-th digit in the binary expansion of this number is 0 or 1, and
  - the fact that the n-th digit is 1 is a mathematical statement that can be formalized in ST.
- Each statement from ST is a finite combination of symbols.
- So in a computer, it is represented as a sequence of 0s and 1s.

16. Levin's Formalization (cont-d)

- This sequence, in its turn, can be interpreted as a natural number.
- Moreover, it is easy to check:
  - whether a given natural number – i.e., in effect, a given sequence of 0s and 1s
  - is a syntactically correct statement of set theory.
- Thus:
  - by trying all possible natural numbers 0, 1, 2, ..., and
  - checking whether each of them is a syntactically correct statement of set theory,
  - we can enumerate all possible such statements into a sequence $S_1, S_2, \ldots$

17. Levin's Formalization (cont-d)

- In these terms, perfect mathematical knowledge can be represented by an infinite binary sequence $\alpha = \alpha_1\alpha_2\ldots$
- Here $\alpha_i = 1$ if and only if the i-th statement α_i is true.
- Similarly, all possible observation and measurement results can also be placed in a single binary sequence.
- Indeed, each observation and measurement result has to be represented in a computer.
- It can, thus, be naturally represented as a sequence of 0s and 1s.
- Each measurement result can thus be described as a sequence of bits.
- Descriptions of the experimental and observational settings (*metadata*) can also be described in a computer.

18. Levin's Formalization (cont-d)

- So, for each bit of each measurement result, we have a description, e.g.,

  "the 5-th bit of measuring wind speed at UTEP campus at 12 pm on August 27, 2019".

- We can similarly sort such descriptions, and get a potentially infinite binary sequence $\omega = \omega_1\omega_1 \ldots$
- Of course, this sequence is only *potentially* infinite.
- At any given moment of time:
  - we have only finitely many measurement and observation results, and
  - thus, we only know a finite part of this potentially infinite sequence.

19. Levin's Formalization (cont-d)

- Levin's formalization of the above commonsense idea is that for every m and n:
  - using the first m bits of the measurement-results sequence $\omega_{1:m} \stackrel{\text{def}}{=} \omega_1 \ldots \omega_m$
  - does not help us compute the truth values $\alpha_{1:n} \stackrel{\text{def}}{=} \alpha_1 \ldots \alpha_n$ of the first n statements of set theory.
- We already know how to formalize this "inability to help".
- Thus, we get the following precise description.
- There exists a small integer $c > 0$ for which, for all m and n, we have $I(\alpha_{1:n} : \omega_{1:m}) \leq c$.
- Levin called this formalization the *Independence Postulate.*

20. This Formalization Is in Perfect Accordance with Modern Physics

- According to modern (quantum) physics, the system's state is described by a wave function.
- We have deterministic equations describing the dynamics of the wave function.
- The state determines the probability of different measurement results.
- The actual sequence of measurement results is random with respect to the corresponding probability measure.
- For such sequence ω, Levin's Independence Postulate is indeed true.
- In this sense, Levin's Independence Postulate is in perfect accordance with modern physics.

21. Every Theory Is Approximate

- Many physicists have yet another intuition, that:
  - no matter what theory we formulate,
  - no matter how well this theory describes the current experimental results,
  - this theory will eventually turn out to be only a good approximation,
  - there will be new experiments, new data that will require a modification of this theory.
- This happened with Newton's mechanics.
- It needed to be modified to take into account relativistic and quantum effects.
- This will happen – many physicists believe – with modern relativistic quantum physics as well.

22. Formalizing This Belief

- In general, the above belief means that:
  - whatever physical law we come up with which is consistent with all physical experiments and observations so far,
  - eventually we will come up with experimental data that violates this law.
- In terms of our notations:
  - currently available results of experiments and observations
  - simply form an initial fragment $\omega_{1:n}$ of the potentially infinite sequence ω of all such results.

23. Formalizing This Belief (cont-d)

- From the mathematical viewpoint, a physical law is simply a property $P(\omega_{1:n})$ that:
  - limits possible values of such fragments
  - to those that satisfy this property.
- Thus, the above physicists' belief is that:
  - for each such property P, there exists an integer M – corresponding to some future moment of time,
  - at which the fragment $\omega_{1:M}$ will *not* satisfy the corresponding property P.

24. Resulting Challenge

- In particular, the physicists' belief means that:
  - no matter what constant c we select in our description of Levin's Independence Principle,
  - there will be a value M for which this principle will be violated, i.e., for which we will have

$$I(\alpha_{1:n}, \omega_{1:M}) > c.$$

- So, contrary to the physicists' intuition (and to modern physics):
  - under this belief,
  - a sequence of observations can't be random.

25. Resulting Challenge (cont-d)

- In other words, the two physicists' intuitions are not fully compatible:
  - the intuition about randomness and
  - the intuition about infinite progress of physics.
- How can we reconcile these two intuitions?

26. How to Reconcile the Two Intuitions: Suggestion And Challenges

- Due to the second (progress-of-science) intuition:
  - we cannot require – as Levine did,
  - that all the values of the information $I(\alpha_{1:n}, \omega_{1:m})$ are bounded by a constant.
- However, intuitively, the first (randomness) intuition tell us that:
  - we cannot expect too much information above complex statements
  - by simply looking at nature.
- We cannot require that the amount of information $I(\alpha_{1:n}, \omega_{1:m})$ be bounded.
- But we *can* require that this amount should be *small* – i.e., it should not grow too fast with m.

27. How to Reconcile the Intuitions (cont-d)

- This idea is informal, can we formalize it?
- Unfortunately, not really; if we:
  - select some slowly growing function $c(m)$ and require that $I(\alpha_{1:n}, \omega_{1:m}) \le c(m)$,
  - we will have the same problem as with the original Levin's Independence Postulate
  - that, according to the progress-of-science intuition, this inequality will be violated for some M.
- Thus, the only way to reconcile the two intuitions is to make an informal statement.
- Thus, *there are fundamental reasons why informal knowledge is needed for describing the real world.*

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