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How Much For a Set: General Case of Decision Making Under Set-Valued Uncertainty

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1. We Need To Make Decisions

- In real life, all the time, we need to make decisions.
- In many cases, decisions are made based on financial consequences:
  - we want to select an investment strategy that will lead us to the largest gain,
  - the company wants to select the design of a new plant so as to get the largest possible profit, etc.
- In the ideal case, we know the exact consequences of each possible alternative.
- In this case, decision is easy: we select the alternative for which the expected gain is the largest.
- In practice, we rarely know the exact consequences of each possible alternative.

2. We Need To Make Decisions (cont-d)

- This happens, e.g., when we invest the money into the government bonds.
- However, this is usually not the most profitable investment.
- In general, we only know these consequences with some uncertainty.
- In other words, for each alternative:
  - instead of a single monetary value of the corresponding financial consequences,
  - we have a whole *set* of possible values.

3. We Need To Make Decisions (cont-d)

- This set S is usually bounded:
  - there is an upper bound on possible gains, and
  - there is an upper bound on possible losses – which means there is a lower bound on possible gains.
- We need to select an alternative based on such bounded sets.

4. Decision-Making Under Set-Valued Uncertainty Reduces to Numerical Estimates

- The simplest situation is when we have only two alternatives:
  - for one of them we know the exact monetary value v, and
  - for the other one, we have a set S of possible values.
- In this case, we need to decide which of these two alternatives is better.
- A practical example can be deciding:
  - whether to invest in government bonds – for which the future monetary value is well known,
  - or to invest in a more volatile financial instrument such as stocks.

5. Reducing to Numerical Estimates (cont-d)

- If the value v is very small, then clearly the set-valued alternative is better.
- We will denote this preference by $v < S$.
- When the value v is very large, then clearly this numerical value is better: $S < v$.
- One can see that there exists a threshold value $v(S)$:
  - the supremum of all the values v for which $v < S$ and
  - the infimum of all the values v for which $S < v$.
- For this threshold:
  - for all $v < v(S)$, we have $v < S$, and
  - for all $v > v(S)$, we have $S < v$.
- So, for any positive number $\varepsilon > 0$, no matter how small it is, we have $v(S) - \varepsilon < S < v(S) + \varepsilon$.

6. Reducing to Numerical Estimates (cont-d)

- When the value ε is sufficiently small, the users cannot distinguish between $v(S)-\varepsilon$, $v(S)$, and $v(S)+\varepsilon$.
- In this sense, the set S is equivalent to the numerical value $v(S)$.
- We will denote this by $S \equiv v(S)$.
- This way, for each set, we get an equivalent numerical value – a "price" for this set.
- Once we know such prices, it is easy to make a decision in the general case of set-valued uncertainty.
- If we have alternatives described by sets $S_1, \ldots, S_n$ then,
  - since each set S_i is equivalent to its price $v(S_i)$,
  - we simply select the alternative i with the largest possible price $v(S_i)$.

7. Reducing to Numerical Estimates (cont-d)

- So:
  - to make decisions under set-values uncertainty,
  - we need to be able to assign a price $v(S)$ to each bounded set.
- In this talk, we explain how to do it.

8. Reasonable Properties of the Desired Price Function $v(S)$

- Let us use common sense to come up with reasonable properties of the desired price function $v(S)$.
- First:
  - if some number r is smaller than the greatest lower bound $\inf S$ of the set S,
  - this implies that this number is smaller than all numbers $s \in S$.
- Here, the set S is the set of all possible gains.
- So the above condition means that the gain r is worse than all possible gains $s \in S$.
- Thus, r is worse than the whole set-valued alternative S:

$$r < S.$$

9. Reasonable Properties of $v(S)$ (cont-d)

- In view of our price-based description of preference between the alternatives:

  the condition $r < S$ is equivalent to $r < v(S)$.

- Thus, if $r < \inf(S)$, then we have $r < v(S)$.
- In particular, for any $\varepsilon > 0$, we have $\inf S - \varepsilon < \inf S$.
- Thus we have $\inf S - \varepsilon < v(S)$.
- In the limit $\varepsilon \to 0$, this implies that $\inf S \le v(S)$.
- Similarly:
  - if some number r is larger than the least upper bound $\sup S$ of the set S,
  - this implies that this number is larger than all numbers $s \in S$.
- Here, the set S is the set of all possible gains.

10. Reasonable Properties of $v(S)$ (cont-d)

- So the above condition means that the gain r is better than all possible gains $s \in S$.
- So, r is better than the whole set-valued alternative S:

$$S < r.$$

- In view of our price-based description of preference between the alternatives:

$$\text{the condition } S < r \text{ is equivalent to } v(S) < r.$$

- Hence, if $\sup(S) < r$, then we have $v(S) < r$.
- In particular, for any $\varepsilon > 0$, we have $\sup S < \sup S + \varepsilon$.
- So we have $v(S) < \sup S + \varepsilon < v(S)$.
- In the limit $\varepsilon \to 0$, this implies that $v(S) \le \sup S$.
- Hence, in general, we have

$$\inf S \le v(S) \le \sup S.$$

11. Reasonable Properties of $v(S)$ (cont-d)

- Another property comes from the fact that usually, we make several different decisions.
- Suppose that we make two consequent independent decisions.
- Let us consider two alternatives corresponding to these two decisions.
- Let us denote:
  - the set of possible gains from the first alternative by S_1, and
  - the set of all possible gains from the second alternative by S_2.

12. Reasonable Properties of $v(S)$ (cont-d)

- Then, by definition of the price function, for us:
  - the first alternative is equivalent to the numerical value $v(S_1)$, and
  - the second alternative is equivalent to the numerical value $v(S_2)$.
- Thus, overall, for these two alternatives, we are willing to pay the price $v(S_1) + v(S_2)$.
- We can view this situation differently:
  - as a single complex alternative in which
  - the set of possible gains is the set of all possible sums $s_1 + s_2$, where $s_1 \in S_1$ and $s_2 \in S_2$.
- The set of all such sums is known as the *Minkowski sum* of the two sets; t is denoted by $S_1 + S_2$:

$$S_1 + S_2 \stackrel{\text{def}}{=} \{s_1 + s_2 : s_1 \in S_1 \,\&\, s_2 \in S_2\}.$$

13. Reasonable Properties of $v(S)$ (cont-d)

- For this set $S_1 + S_2$, by definition of the price function, we pay the price $v(S_1 + S_2)$.
- However, this is the exact same situation as before, when we estimated the price of this situation as
$$v(S_1) + v(S_2).$$
- So, these two estimates must coincide:
$$v(S_1 + S_2) = v(S_1) + v(S_2).$$

14. Closed Sets

- A set S is called *closed* if:
  - for every sequence of elements $s_n \in S$ from this set that converges to some value s,
  - the limit point s also belongs to the set S.
- One can easily check that a closed interval $[a, b] \stackrel{\text{def}}{=} \{x : a \le x \le b\}$ is a closed set.
- On the other hand, an open interval $(a, b) \stackrel{\text{def}}{=} \{x : a < x < b\}$ is *not* a closed set:
  - a sequence of elements $s_n = a + 2^{-n}$ belongs to the open interval, but
  - its limit a does not.

15. Closed Sets (cont-d)

- *Known Result:* let $v(S)$ assigns, to each bounded closed set S, a numerical value so that:
  - if $\inf S \le v(S) \le \sup S$ and $v(S_1 + S_2) = v(S_1) + v(S_2)$ for S and S_i,
  - then $v(S) = \alpha \cdot \sup S + (1 - \alpha) \cdot \inf S$ for some $\alpha \in [0, 1]$.
- This result has been extended to fuzzy uncertainty with continuous membership functions.
- Indeed, for these functions, α-cuts are closed sets.
- We extend this result to general bounded sets.
- We hope that:
  - our new results can be extended to fuzzy sets
  - for which the membership function can be discontinuous.

16. Why Do We Need to Go Beyond Closed Sets

- Suppose that we are trying to sell something that is worth d dollars.
- However, by law, one need to have a license to sell this object.
- So, the only way to sell is to hire someone with a license who will sell it for us.
- Of course, we need to pay something to that licensed person.
- As a result what we get is smaller than d.
- We can imagine situations in which there is a competition between licensed sellers.
- We will then be able to make this intermediary's cost as little as possible.
- However, we can never make it 0.

17. Need to Go Beyond Closed Sets (cont-d)

- Thus, we may be able to get gains close to d, but never exactly d.
- Similarly, we can encounter situations in which licensed sellers are rare.
- Then, they can charge as much as possible – all the way to the cost d.
- They may charge 90% of the value d, but they cannot charge 100%.
- Indeed, for 100%, there is no sense for us to sell at all.
- Thus, our gain can be close to 0, but never exactly equal to 0.
- In this case, the set of possible gains is the open interval $(0, d)$, which is *not* a closed set.

18. Our Main Result

- By a *price function*, we mean a function $v(S)$ that assigns, to each bounded set S, a number $v(S)$ so that:
  - for every set S, we have $\inf S \le v(S) \le \sup S$ (*monotonicity*), and
  - for every two sets S_1 and S_2, we have $v(S_1 + S_2) = v(S_1) + v(S_2)$ (*additivity*).
- Our Main Result:
  - Every price function has the form

    $v(S) = \alpha \cdot \sup S + (1 - \alpha) \cdot \inf S$ for some $\alpha \in [0, 1]$.
  - Vice versa, for each $\alpha \in [0, 1]$, the mapping $v(S) = \alpha \cdot \sup S + (1 - \alpha) \cdot \inf S$ is a price function.

19. Proof

- Due the known result, for closed sets, $v(S) = \alpha \cdot \sup S + (1-\alpha) \cdot \inf S$ for some $\alpha \in [0,1]$.
- Let us now consider the simplest case of non-closed sets that we have described earlier:
  - open intervals (a,b) and
  - half-open intervals $(a,b]$ and $[a,b)$.
- One can check that for each $a < b$, we have

$$(a,b)+(a,b)=(2a,2b), \quad (a,b)+[a,b]=(2a,2b),$$

$$(a,b)+(a,b]=(2a,2b), \quad (a,b)+[a,b)=(2a,2b).$$

- Indeed, let us prove that $(a,b)+(a,b)=(2a,2b)$.
- If $x \in (a,b)$ and $y \in (a,b)$, then $a < x < b$ and

$$a < y < b.$$

20. Proof (cont-d)

- By adding these two inequalities, we conclude that $2a < x + y < 2b$, i.e., $x + y \in (2a, 2b)$.
- Vice versa, let $z \in (2a, 2b)$.
- Then, from $2a < z < 2b$, we conclude that $a < \dfrac{z}{2} < b$.
- Thus z can be represented as the sum of two values $z = \dfrac{z}{2} + \dfrac{z}{2}$ from the interval (a, b).
- Similarly, we can prove three other equalities.
- Due to additivity, from $(a, b) + (a, b) = (2a, 2b) = (a, b) + [a, b] = (2a, 2b)$, we conclude that

$$v((a, b)) + v((a, b)) = v((2a, 2b)) = v((a, b)) + v([a, b]),$$

$$\text{so } v((a, b)) = v([a, b]).$$

- Similarly, from other equalities, we conclude that $v((a, b]) = v([a, b])$ and $v([a, b)) = v([a, b])$.

21. Proof (cont-d)

- A closed interval is a closed set, so

$$v([a,b]) = \alpha \cdot b + (1-\alpha)\cdot a, \text{ thus}$$

$$v((a,b)) = v((a,b]) = v([a,b)) = v([a,b]) = \alpha\cdot b+(1-\alpha)\cdot a.$$

- So, for open and half-open intervals, we have the desired property

$$v((a,b)) = v((a,b]) = v([a,b)) = v([a,b]) =$$

$$\alpha \cdot \sup(a,b) + (1-\alpha)\cdot \inf(a,b).$$

- Let us now consider a generic set S.
- We want to prove that the price of this set is equal to the desired convex combination of the value

$$a \stackrel{\text{def}}{=} \inf S \text{ and } b \stackrel{\text{def}}{=} \sup S.$$

- Our proof will depend on whether these values $\inf S$ and $\sup S$ belong to the set S or not.

22. Proof (cont-d)

- Thus, we will consider four possible cases:
  - the case when $\inf S \in S$ and $\sup S \in S$;
  - the case when $\inf S \in S$ and $\sup S \notin S$;
  - the case when $\inf S \notin S$ and $\sup S \in S$;
  - the case when $\inf S \notin S$ and $\sup S \notin S$.
- Let us consider these four cases one by one.

23. First Case

- Let us first consider the case when $a \in S$ and $b \in S$.
- Let us prove that in this case, we have $S + [a, b] = [2a, 2b]$, i.e., that:
  - every element of the set $S + [a, b]$ belongs to the set $[2a, 2b]$, and
  - vice versa, every element of the set $[2a, 2b]$ belongs to the set $S + [a, b]$.
- Indeed, by definition of infimum and supremum, for every $x \in S$, we have $a \le x \le b$.
- When $y \in [a, b]$, we also have $a \le y \le b$.
- Adding these inequalities, we get $2a \le x + y \le 2b$, i.e., $x + y \in [2a, 2b]$.
- Vice versa, let $z \in [2a, 2b]$.
- If $2a \le z \le a + b$, then $a \le z - a \le b$, so $z - a \in [a, b]$.

24. First Case (cont-d)

- Thus, we can represent z as the sum $a + (z - a)$, where $a \in S$ and $z - a \in [a, b]$.
- Similarly, if $a + b \leq z \leq 2b$, then $a \leq z - b \leq b$.
- Thus, we can represent z as the sum $b + (z - b)$, where $b \in S$ and $z - b \in [a, b]$.
- In both case, $z \in S + [a, b]$.
- The equality $S + [a, b] = [2a, 2b]$ is thus proven.
- By using additivity, we conclude that $v(S) + v([a, b]) = v([2a, 2b])$ hence $v(S) = v([2a, 2b]) - v([a, b])$.
- Substituting the known expression for the prices of the intervals, we get the desired equality

$$v(S) = \alpha \cdot b + (1 - \alpha) \cdot a.$$

25. Second Case: $a \in S$ and $b \notin S$

- Let us show that in this case, $S + [a, b] = [2a, 2b)$.
- Indeed, by definition of infimum and supremum, for every $x \in S$, we have $a \le x \le b$.
- Since $b = \sup S$ is not attained: $a \le x < b$.
- When $y \in [a, b]$, we have $a \le\le b$. Adding these inequalities, we get $2a \le x + y < 2b$, i.e., $x + y \in [2a, 2b)$.
- Vice versa, let $z \in [2a, 2b)$.
- If $2a \le z \le a + b$, then $a \le z - a \le b$, so $z - a \in [a, b]$.
- Thus, we can represent z as the sum $a + (z - a)$, where $a \in S$ and $z - a \in [a, b]$.

26. Second Case (cont-d)

- If $a + b \le z < 2b$, then $a \le z - b < b$.
- Since b is the supremum of the set S, by definition of the supremum, there exists a point $z' \in S$ for which

$$a \le z - b \le z' \le b.$$

- Let us show that the difference $z - z'$ belongs to the interval $[a, b]$.
- Indeed, from $z - b \le z'$, we conclude that $z - z' \le b$.
- From the fact that $a+b \le z$ and $z' \le b$ thus $-b \le -z'$, we conclude that $(a + b) + (-b) \le z + (-z')$, i.e., that

$$a \le z - z'.$$

- So, the value z can be represented as $z'+(z-z')$, where $z' \in S$ and $z - z' \in [a, b]$.
- The equality $S + [a, b] = [2a, 2b)$ is thus proven.

27. Second Case (cont-d)

- The equality $S + [a, b] = [2a, 2b)$ is proven.
- By using additivity, we conclude that $v(S)+v([a, b]) = v([2a, 2b))$ hence $v(S) = v([2a, 2b)) - v([a, b])$.
- Substituting the expression for the prices of the intervals, we get the desired equality $v(S) = \alpha\cdot b+(1-\alpha)\cdot a$.

28. Third and Fouth Cases

- Similar arguments show that:
  - when $a \notin S$ and $b \in S$, then $S + [a, b] = (2a, 2b]$ and thus, $v(S) = \alpha \cdot b + (1 - \alpha) \cdot a$; and
  - when $a \notin S$ and $b \notin S$, then $S + [a, b] = (2a, 2b)$ and thus, too $v(S) = \alpha \cdot b + (1 - \alpha) \cdot a$.
- Our main result is proven.

29. What If We Do Not Require Additivity?

- What if we do no require additivity.
- For example, we may be talking about preferences in general, not necessarily monetary preferences.
- In this case, as we will show, we can still reduce a general set to the case of an interval.
- If $a < b$, then clearly b is better.
- For the case of uncertainty, it may be that:
  - each alternative a from the set A is smaller then or equal to some alternative b from the set B, and
  - each alternative b from the set B is larger than or equal to some alternative a from the set A
- Then it is reasonable to conclude that B is better or of the same quality as A; we will denote it by $A \le B$.
- If $A \le B$ and $B \le A$, we will denote it by $A \equiv B$.

30. No-Additivity Case (cont-d)

- Let us describe it in precise terms.
- Let $\leq$ be a reflexive ($A \leq A$) and transitive relation on the class of all bounded subsets of the real line.
- We say that this relation is *monotonic* if $A \leq B$ when the following two properties hold:
  - for every $a \in A$, there exists a $b \in B$ for which $a \leq b$, and
  - for every $b \in B$, there exists an $a \in A$ for which $a \leq b$.

31. Second Result

- For every monotonic relation, each bounded set is equivalent to an interval:
  - if $\inf S \in S$ and $\sup S \in S$, then $S \equiv [\inf S, \sup S]$;
  - if $\inf S \in S$ and $\sup S \notin S$, then $S \equiv [\inf S, \sup S)$;
  - if $\inf S \notin S$ and $\sup S \in S$, then $S \equiv (\inf S, \sup S]$;
  - if $\inf S \notin S$ and $\sup S \notin S$, then $S \equiv (\inf S, \sup S)$.

32. Proof: Idea

- We will consider the four cases one by one.
- Let us denote $s_- \stackrel{\text{def}}{=} \inf S$ and $s_+ \stackrel{\text{def}}{=} \sup S$.

33. First Case

- Let us first consider the case when $s_- \in S$ and $s_+ \in S$.
- We want to prove that in this case, we have
$$S \equiv [s_-, s_+].$$
- For this, we need to prove that $S \le [s_-, s_+]$ and that $[s_-, s_+] \le S$.
- Let us first prove that $S \le [s_-, s_+]$;
- For each $s \in S$, by definition of supremum, we have
$$s \le s_+.$$
- Since $s_+ \in [s_-, s_+]$, the first property from the definition of monotonicity is satisfied.
- Similarly, for each $s \in [s_-, s_+]$, we have $s_- \le s$.
- Since $s_- \in S$, the second property is also satisfied.
- Thus, indeed, $S \le [s_-, s_+]$.

34. First Case (cont-d)

- Let us now prove that $[s_-, s_+] \leq S$.
- For each $s \in [s_-, s_+]$, we have $s \leq s_+$.
- Since $s_+ \in S$, the first property is satisfied.
- Similarly, for each $s \in S$, we have $s_- \leq s$.
- Since $s_- \in [s_-, s_+]$, the 2nd property is also satisfied.
- Thus, indeed, $[s_-, s_+] \leq S$, thus $S \equiv [s_-, s_+]$.

35. Second Case: $s_- \in S$ and $s_+ \notin S$

- We want to prove that in this case, $S \equiv [s_-, s_+)$.
- To prove thus, we need to prove that $S \leq [s_-, s_+)$ and that $[s_-, s_+) \leq S$.
- Let us first prove that $S \leq [s_-, s_+)$;
- For each $s \in S$, by definition of supremum, we have $s \leq s_+$.
- Since the supremum is not attained, $s < s_+$.
- Thus, we have $s \leq s$ for $s \in [s_-, s+)$.
- So, the first property is satisfied.
- Let $s \in [s_-, s_+)$, then $s_- \leq s$.
- Since $s_- \in S$, the second property is also satisfied.

36. Second Case (cont-d)

- Thus, indeed, $S \leq [s_-, s_+)$.
- Let us now prove that $[s_-, s_+) \leq S$; indeed:
- For each $s \in [s_-, s_+)$, we have $s < s_+$.
- s_+ is the supremum of the set S – i.e., the least upper bound.
- Since the value s is smaller that s_+, this means that s is not the upper bound for the set S.
- In other words, this means that there exists a value $s' \in S$ which is not smaller than or equal to s.
- So, s' is larger than s: $s < s'$.
- So, the first property is satisfied.

37. Second Case (cont-d)

- Now, for each $s \in S$, we have $s_- \le s$.
- Since $s_- \in [s_-, s_+]$, the second property is also satisfied.
- Thus, indeed, $[s_-, s_+) \le S$, and $S \equiv [s_-, s_+)$.
- Similarly, we can prove the equivalence for two other cases.
- The second result is proven.

38. Relation Between Open, Closed, And Half-Open Intervals

- Let us analyze the relation between open, closed, and half-open intervals.
- For this, let us take into account that:
  - if $\varepsilon > 0$ is sufficiently small,
  - then we do not see the difference between ε-closed values.
- We say that a reflexive and transitive relation $\leq$ between bounded sets of is *realistic* if $A \equiv B$ when:
  - for every $a \in A$ and for every $\varepsilon > 0$, there exists a value $b \in B$ which is ε-close to a; and
  - for every $b \in B$ and for every $\varepsilon > 0$, there exists a value $a \in A$ which is ε-close to b.

39. Third Result

- For each realistic relation, for all $a < b$, we have

$$[a,b] \equiv (a,b) \equiv (a,b] \equiv [a,b).$$

- Thus, intervals of different type are equivalent to each other – provided that their endpoints coincide.
- For monotonic relations:
  - in view of the second result,
  - every bounded set S is equivalent to an interval.
- Thus, if this relation is also realistic, we always have

$$S \equiv [\inf S, \sup S].$$

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