Complex-Valued Interval Computations Are NP-Hard Even for Single Use Expressions

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1. Need for hypothesis testing and for interval computations

- In many practical situations:
  - the value y of a physical quantity depends on the values $x_1, \ldots, x_n$ of related quantities,
  - but we do not know the exact form of this dependence.
- In such cases, based on the available observation, researchers come up with a hypothesis that $y = f(x_1, \ldots, x_n)$ for some specific function f.
- How can we test this hypothesis?
- A natural idea is to consider other situations in which we know both the value y and the values $x_1, \ldots, x_n$.
- In the ideal situation, in which we know the exact values of y and x_i, this testing is simple.
- We simply check whether for these measurement results, y is equal to $f(x_1, \ldots, x_n)$.
- In practice, however, measurements are never absolutely accurate.

2. Need for hypothesis testing and for interval computations (cont-d)

- For each quantity q, the measurement result $\widetilde{q}$ is, in general, different from the actual (unknown) value q of this quantity.
- In many practical situations:
  - the only information that we have about the difference $\Delta q \stackrel{\text{def}}{=} \widetilde{q} - q$ (known as the *measurement error*)
  - is the upper bound Δ on its absolute value: $|\Delta q| \le \Delta$.
- In this case, based on the measurement result $\widetilde{q}$:
  - the only information that we have about the actual value q
  - is that this value is contained in the interval $[\widetilde{q} - \Delta, \widetilde{q} + \Delta]$.
- Under such interval uncertainty, after the measurement, we have:
  - the interval $[\underline{y}, \overline{y}]$ of possible values of y and
  - the intervals $[\underline{x}_i, \overline{x}_i]$ of possible values of each quantity x_i.

3. Need for hypothesis testing and for interval computations (cont-d)

- In this case, the original hypothesis is confirmed if there exist values within these intervals for which $y = f(x_1, \ldots, x_n)$.
- In other words, the hypothesis is confirmed if the interval $[\underline{y}, \overline{y}]$ has a common point with the range

$$f([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \stackrel{\text{def}}{=}$$

$$\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

- A natural way to check this is to compute the range and then to check whether the intersection of the range and the y-interval is non-empty.
- Computing the range is known as *interval computations*.

4. Comments

- The measurement results $\widetilde{q}$ are usually rational numbers.
- In modern computer age, they come from a computer and are, thus, binary rational, i.e., number of the type $m/2^n$ for integers m and n.
- Similarly, the bounds Δ on the absolute value of the measurement error are rational.
- Because of this, the endpoints of the resulting interval $[\widetilde{q}-\Delta, \widetilde{q}+\Delta]$ are usually also rational numbers.
- In the case when we have *fuzzy* information about y and x_i, we have *fuzzy* sets corresponding to y and $f(x_1, \ldots, x_n)$.
- In this case, a natural idea it to look for the largest α for which the α-cuts of these two sets have a common point.
- This is the degree to which the given fuzzy data is consistent with the proposed hypothesis.

5. Comments (cont-d)

- It is known that the α-cut of the fuzzy set $f(x_1, \ldots, x_n)$ is equal to the range (1) of the function f on the α-cuts of x_i.
- Thus, from the computational viewpoint, the fuzzy version of this problem can be reduced to its interval version.

6. Computational complexity of interval computations

- It is known that the problem of computing the interval range is, in general, NP-hard.
- It is NP-hard already for quadratic functions $f(x_1, \ldots, x_n)$.
- However, there are classes of expressions for which computing the range is feasible. First, these are functions representing elementary arithmetic operations.
- In these cases, the resulting expressions come from the fact that the corresponding functions are monotonic.
- The range for the sum $f(x_1, x_2) = x_1 + x_2$ is equal to $[\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2]$.
- The range for the difference $f(x_1, x_2) = x_1 - x_2$ is equal to

$$[\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2].$$

7. Comput. complexity of interval computations (cont-d)

- The range for the product $f(x_1, x_2) = x_1 \cdot x_2$ is equal to $[\underline{y}, \overline{y}]$, where

$$\underline{y} = \min(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \overline{x}_2, \overline{x}_1 \cdot \underline{x}_2, \overline{x}_2 \cdot \overline{x}_2),$$

$$\overline{y} = \max(\underline{x}_1 \cdot \underline{x}_2, \underline{x}_1 \cdot \overline{x}_2, \overline{x}_1 \cdot \underline{x}_2, \overline{x}_2 \cdot \overline{x}_2).$$

- The range for the inverse $f(x_1) = 1/x_1$ is equal to $[1/\overline{x}_1, 1/\underline{x}_1]$ if $0 \notin [\underline{x}_1, \overline{x}_1]$.
- Similarly, we can derive explicit formulas for the ranges of elementary functions.
- For example, for $f(x_1) = x_1^2$, the range is equal:
  - to $[\underline{x}_1^2, \overline{x}_1^2]$ if $0 \le \underline{x}_1$;
  - to $[\overline{x}_1^2, \underline{x}_1^2]$ if $\overline{x}_1 \le 0$;
  - to $[0, \max(\underline{x}_1^2, \overline{x}_1^2)]$ if $\underline{x}_1 \le 0 \le \overline{x}_1$.
- These formulas form *interval arithmetic.*
- The range of the sum is called the sum of the intervals.

8. Comput. complexity of interval computations (cont-d)

- The range of the difference is called the difference between the intervals.
- The range of the square is called the square of the interval, etc.
- Another example when there is a feasible algorithm for computing the range is the so-called *single use expressions* (SUE, for short).
- These are expressions in which each variable occurs only once.
- Examples of such expressions include:
  - the product $x_1 \cdot \ldots \cdot x_n$,
  - the value $(1/n) \cdot \sum\limits_{i=1}^{n} x_i^2$, etc.

9. Comput. complexity of interval computations (cont-d)

- Such expressions are ubiquitous in physics:
  - Ohm's law $V = I \cdot R$,
  - formula for the kinetic energy $E = (1/2) \cdot m \cdot v^2$,
  - formula for the gravitational force $F = G \cdot m_1 \cdot m_2 \cdot r^{-2}$, etc.
- Of course, there are expressions of the type $y = x - x^2$ in which the variable x occurs twice.
- For single-use expressions, the interval range can obtained if we simply:
  - replace each arithmetic operation (that form the computation algorithm)
  - with the corresponding operation of interval arithmetic.

10. Need for complex values and for the corresponding hypothesis testing and interval computations

- Many physical quantities are complex-valued:
  - complex amplitude and impedance in electrical engineering,
  - complex wave function in quantum mechanics, etc.
- Similarly to the real-valued case, we can have a complex-valued quantity z depend on the complex-values quantities $z_1, \ldots, z_n$;so:
  - we can have hypotheses $z = f(z_1, \ldots, z_n)$, and
  - we need to check whether the new measurement results are consistent with these hypotheses.
- In many such situations:
  - to measure the corresponding complex value $z = x + \mathrm{i} \cdot y$,
  - we separately measure the real part x and the imaginary part y.
- After the measurement, we get the interval $[\underline{x}, \overline{x}]$ of possible values of x and the interval $[\underline{y}, \overline{y}]$ of possible values of y.

11. Need for complex values and for the corresponding hypothesis testing and interval computations (cont-d)

- So, we get the set of possible complex numbers

$$\mathbf{z} \stackrel{\text{def}}{=} \{x + \mathrm{i} \cdot y : x \in [\underline{x}, \overline{x}], y \in [\underline{y}, \overline{y}]\}.$$

- It is reasonable to call this set a *complex interval.*
- Based on these interval-valued measurement results, we need to check whether the complex interval for z has a common point with the range

$$f(\mathbf{z}_1, \ldots, \mathbf{z}_n) \stackrel{\text{def}}{=} \{f(z_1, \ldots, z_n) : z_1 \in \mathbf{z}_1, \ldots, z_n \in \mathbf{z}_n\}.$$

12. Computational complexity of complex interval computations: what is known and what we want to analyze

- Of course, real-valued computations can be viewed as a particular case of complex-values ones.
- Indeed, it is sufficient to take all imaginary parts equal to 0.
- Since real-valued interval computations are NP-hard, this implies that complex-valued interval computations are NP-hard as well.
- A natural question is: what about the SUE expressions?
- In this paper, we show that, in contrast to real-valued case, complex interval arithmetic is NP-hard even for SUE expressions.

13. Main Result

- Let $\underline{x}$, $\overline{x}$, $\underline{y}$, and $\overline{y}$ be rational numbers.
- We then call the following set a *complex interval*:

$$\mathbf{z} = [\underline{x}, \overline{x}] + \mathrm{i} \cdot [\underline{y}, \overline{y}] = \{x + \mathrm{i} \cdot y : x \in [\underline{x}, \overline{x}] \text{ and } y \in [\underline{y}, \overline{y}]\}.$$

- Let $z = f(z_1, \ldots, z_n)$ be a complex-valued function.
- By a *problem of complex interval computations* we mean the following problem:
  - given: complex intervals $\mathbf{z}$, $\mathbf{z}_1$, ..., $\mathbf{z}_n$,
  - check whether the complex interval $\mathbf{z}$ and the range $f(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ have a common point.

14. Main Result (cont-d)

- **Main result.** *For each of the following SUE functions, the problem of complex interval computations is NP-hard:*

  1. *the scalar (dot) product*

  $$f(z_1, \ldots, z_n, t_1, \ldots, t_n) = \sum_{i=1}^{n} z_i \cdot t_i;$$

  2. *the second moment*

  $$f(z_1, \ldots, z_n) = \frac{1}{n} \cdot \sum_{i=1}^{n} z_i^2;$$

  3. *the product* $f(z_1, \ldots, z_n) = z_1 \cdot \ldots \cdot z_n$.

15. Proof: general idea

- To prove NP-hardness of all three range computation problems, we will reduce,
  - to this new problem,
  - a known NP-hard *partition problem*.
- The partition problem is as follows:
  - given n positive integers $s_1, \ldots, s_n$,
  - to check whether there exists values $\varepsilon_i \in \{-1, 1\}$ such that

$$\sum_{i=1}^{m} \varepsilon_i \cdot s_i = 0.$$

16. Proof for the first function

- Let us first prove that the first complex interval computations problem is NP-hard.
- To prove this:
  - for every instance $s_1, \ldots, s_n$ of the partition problem,
  - we take $\mathbf{z}_i = s_i \cdot (1 + \mathrm{i} \cdot [-1, 1])$, $\mathbf{t}_i = 1 + \mathrm{i} \cdot [-1, 1]$ and $\mathbf{z} = [0, 0]$.
- Then, possible values $z_i \in \mathbf{z}_i$ have the form $z_i = s_i \cdot (1 + \mathrm{i} \cdot a_i)$ for some $a_i \in [-1, 1]$.
- Possible values of t_i are of the form $t_i = 1 + \mathrm{i} \cdot b_i$ with $b_i \in [-1, 1]$.

17. Proof for the first function (cont-d)

- Let us prove that:
  - the interval $\mathbf{z}$ has a common point with the range

$$f(\mathbf{z}_1, \ldots, \mathbf{z}_n, \mathbf{t}_1, \ldots, \mathbf{t}_n),$$

  - i.e., equivalently, that the point $z = 0$ belongs to the range of f,
  - if and only if the given instance of the partition problem has a solution.
- Assume first that the partition problem has a solution ε_i for which $\sum s_i \cdot \varepsilon_i = 0$.
- Then, as one can easily check, for $z_i = s_i \cdot (1 + \mathrm{i} \cdot \varepsilon_i)$ and $t_i = 1 + \mathrm{i} \cdot \varepsilon_i$, we have $z_i \cdot t_i = s_i \cdot (1 + 2\mathrm{i} \cdot \varepsilon_i - 1) = 2\mathrm{i} \cdot s_i \cdot \varepsilon_i$, and thus,

$$f(z_1, \ldots, z_n, t_1, \ldots, t_n) = \sum_{i=1}^{n} z_i \cdot t_i = 2\mathrm{i} \cdot \sum_{i=1}^{n} s_i \cdot \varepsilon_i = 0 = z.$$

- So, in this case, the interval $\mathbf{z}$ has a common point with the range $f(\mathbf{z}_1, \ldots, \mathbf{z}_n, \mathbf{t}_1, \ldots, \mathbf{t}_n)$.

18. Proof for the first function (cont-d)

- Let us now prove that, vice versa:
  - if the interval $\mathbf{z}$ has a common point with the range $f(\mathbf{z}_1, \ldots, \mathbf{z}_n, \mathbf{t}_1, \ldots, \mathbf{t}_n)$,
  - i.e., if $0 = f(z_1, \ldots, z_n, t_1, \ldots, t_n)$ for some values $z_i \in \mathbf{z}_i$ and $t_i \in \mathbf{t}_i$,
  - then the corresponding instance of the partition problem has a solution.
- Indeed, for each i, the product $z_i \cdot t_i$ is equal to

$$s_i \cdot (1 + \mathrm{i} \cdot a_i) \cdot (1 + \mathrm{i} \cdot b_i) = s_i \cdot ((1 - a_i \cdot b_i) + \mathrm{i} \cdot (a_i + b_i)).$$

- Since $|a_i| \le 1$ and $|b_i| \le 1$, we have $|a_i \cdot b_i| \le 1$, and therefore, $1 - a_i \cdot b_i \ge 0$.
- Thus, the real part of the sum $\sum\limits_{i=1}^{n} z_i \cdot t_i$ is equal to the sum of n non-negative numbers $s_i \cdot (1 - a_i \cdot b_i)$.

19. Proof for the first function (cont-d)

- The only possibility for this sum to be equal to 0 if when all n non-negative terms are equal to 0, i.e., when $a_i \cdot b_i = 1$.
- Since $|a_i| \le 1$ and $|b_i| \le 1$, the absolute value of the product $|a_i \cdot b_i|$ cannot exceed 1.
- So, the only possibility for this product to be equal to 1 is when both absolute values are equal to 1, i.e., when $a_i = \pm 1$ and $b_i = \pm 1$.
- Since $a_i \cdot b_i = 1$, the signs must coincide, i.e., we must have
$$a_i = b_i \in \{-1, 1\}.$$
- Let us denote the common value of a_i and b_i by ε_i.
- For these values $a_i = b_i = \varepsilon_i$, the imaginary part of $z_i \cdot t_i$ is equal to $2 \cdot \varepsilon_i \cdot s_i$.
- So, the fact that the imaginary part of the sum $\sum\limits_{i=1}^{n} z_i \cdot t_i$ is equal to 0 is equivalent to $2 \cdot \sum\limits_{i=1} \varepsilon_i \cdot s_i = 0$.

20. Proof for the first function (cont-d)

- So, the original instance of the partition problem indeed has a solution.
- The statement is proven.

21. Proof for the second function: case of full squares

- Let us now prove that complex interval computation problem is NP-hard for the second function.
- If all the values s_i are squares of integers, then we can take

$$\mathbf{z}_i = \sqrt{s_i} \cdot (1 + \mathrm{i} \cdot [-1, 1]) \text{ and } \mathbf{z} = [0, 0].$$

- In this case, all possible values of z_i have the form $z_i = \sqrt{s_i} \cdot (1 + \mathrm{i} \cdot a_i)$ for some $a_i \in [-1, 1]$.
- Then, we have $z_i^2 = s_i \cdot ((1 - a_i^2) + \mathrm{i} \cdot (2a_i))$.
- If the corresponding instance of the partition problem has a solution, then, as one can easily check, we have $0 = f(z_1, \ldots, z_n)$ for

$$z_i = \sqrt{s_i} \cdot (1 + \mathrm{i} \cdot \varepsilon_i).$$

- Vice versa, let us assume that $0 = f(z_1, \ldots, z_n)$ for some values

$$z_i = \sqrt{s_i} \cdot (1 + \mathrm{i} \cdot a_i) \in \mathbf{z}_i.$$

- In this case, since $|a_i| \le 1$, we have $1 - a_i^2 \ge 0$.

22. Proof for the second function: case of full squares (cont-d)

- So, the only possibility for the expression $\dfrac{1}{n}\cdot\sum\limits_{i=1}^{n} z_i^2$ to have a zero real part is to have $1-a_i^2=0$ for all i.
- This means $a_i=\pm 1$ for every i.
- In other words, we have $a_i=\varepsilon_i\in\{-1,1\}$.
- For these a_i, the imaginary part of the expression $\dfrac{1}{n}\cdot\sum\limits_{i=1}^{n} z_i^2$ is equal to $\dfrac{2}{n}\cdot\sum\limits_{i=1}^{n} s_i\cdot\varepsilon_i$.

23. Proof for the second function: case of full squares (cont-d)

- Thus, the fact that the imaginary part is equal to 0 is equivalent to

$$\sum_{i=1}^{n} \varepsilon_i \cdot s_i = 0.$$

- This means the existence of the solution to the original instance of the partition problem.

24. Proof for the second function: general case

- When the values s_i are not full squares, then, in defining $\mathbf{z}_i$:
  - instead of $\sqrt{s_i}$,
  - we can take rational numbers r_i for which r_i^2 is ε-close to $\sqrt{s_i}$ for some small $\varepsilon > 0$.
- Instead of $\mathbf{z} = [0, 0]$, we take $\mathbf{z} = \mathrm{i} \cdot [-\delta, \delta]$ for some small $\delta > 0$.
- Let us prove that for appropriately chosen ε and δ:
  - the original instance of the partition problem has a solution
  - if and only if the sets $\mathbf{z}$ and $f(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ have a common point.
- If the corresponding instance of the partition problem has a solution, then we can take $z_i = r_i \cdot (1 + \mathrm{i} \cdot \varepsilon_i)$, in which case

$$\frac{1}{n} \cdot \sum_{i=1}^{n} z_i^2 = \mathrm{i} \cdot \frac{2}{n} \cdot \sum_{i=1}^{n} r_i^2.$$

25. Proof for the second function: general case (cont-d)

- Since each value r_i^2 is ε-close to s_i, we conclude that

$$\left|\frac{2}{n}\cdot\sum_{i=1}^{n} r_i^2 - \frac{2}{n}\cdot\sum_{i=1}^{n} s_i\right| \le \frac{2}{n}\cdot\sum_{i=1}^{n}\varepsilon = \frac{2}{n}\cdot n\cdot\varepsilon = 2\varepsilon.$$

- So, for $2\varepsilon \le \delta$, we get $f(z_1,\ldots,z_n) \in \mathbf{z}$, and thus, the sets $\mathbf{z}$ and $f(\mathbf{z}_1,\ldots,\mathbf{z}_n)$ have a common point.
- Vice versa, let us assume that the sets $\mathbf{z}$ and $f(\mathbf{z}_1,\ldots,\mathbf{z}_n)$ have a common point.
- Let us prove that in this case, the original instance of the partition problem has a solution.
- Indeed, in this case, similarly to exact squares case:
  - from the fact that there is a common point,
  - we can still conclude that for the corresponding common point, we have $a_i = \pm 1$.

26. Proof for the second function: general case (cont-d)

- For these a_i, the imaginary part I of the expression $\dfrac{1}{n}\cdot\sum\limits_{i=1}^{n} z_i^2$ is equal to $\dfrac{2}{n}\cdot\sum\limits_{i=1}^{n} r_i^2\cdot\varepsilon_i$.
- The absolute value of the imaginary part is bounded by δ, so
$$|I| = \left|\frac{1}{n}\cdot\sum_{i=1}^{n}\varepsilon_i\cdot r_i^2\right| \le \delta.$$
- Since each value r_i^2 is ε-close to s_i, we conclude that
$$\left|I - \frac{2}{n}\cdot\sum_{i=1}^{n} s_i\cdot\varepsilon_i\right| \le 2\varepsilon.$$
- Thus,
$$\left|\frac{2}{n}\cdot\sum_{i=1}^{n} s_i\cdot\varepsilon_i\right| \le |I| + \left|I - \frac{2}{n}\cdot\sum_{i=1}^{n} s_i\cdot\varepsilon_i\right| \le 2\varepsilon + \delta.$$

27. Proof for the second function: general case (cont-d)

- Hence $\left|\sum_{i=1}^{n} s_i \cdot \varepsilon_i\right| \le \dfrac{n}{2} \cdot (2\varepsilon + \delta)$.
- The sum $\sum s_i \cdot \varepsilon_i$ is an integer.
- So if $\dfrac{n}{2} \cdot (2\varepsilon + \delta) < 1$, this implies:
  - that $\sum s_i \cdot \varepsilon_i = 0$,
  - i.e., that the values $\varepsilon_i \in \{-1, 1\}$ provide a solution to the original instance of the partition problem.
- By combining the above results, we conclude that the desired equivalence can be proven if we have $2\varepsilon \le \delta$ and $(n/2) \cdot (2\varepsilon + \delta) < 1$.
- These two inequalities are satisfies for sufficiently small ε and δ, e.g., for $\delta = 1/(2n)$ and $\varepsilon = 1/(4n)$.
- The statement is proven.

28. Proof for the third function

- Let us now prove that for the third function, the complex interval computation problem is NP-hard.
- For every instance of the partition problem, we compute

$$k = 1/\left(\sum_{j=1}^{n} s_j\right),\ \theta_i = k \cdot s_i, \text{ and } \tan(\theta_i).$$

- Let us first consider the case when all the values $\tan(\theta_i)$ and the product $\prod_{i=1}^{n} \sqrt{1+\tan^2(\theta_i)}$ are rational numbers.
- In this case, we can take $\mathbf{z}_i = 1 + \mathrm{i} \cdot [-t_i, t_i]$ for $t_i = \tan(\theta_i)$ and

$$z = \prod_{i=1}^{n} \sqrt{1+t_i^2}.$$

29. Proof for the third function (cont-d)

- Let us prove that:
  - the selected number z belongs to the range of the product
  - if and only if the original instance of the partition problem has a solution.
- In this proof, we will use the known fact that every complex number $z = x + \mathrm{i} \cdot y$ can be represented in a polar form $z = \rho \cdot e^{\mathrm{i} \cdot \alpha}$.
- Here, $\rho = \sqrt{x^2 + y^2}$ is the absolute value (magnitude) of z.
- The "phase" θ is the angle between the direction from 0 to z and the positive real semi-axis.
- When we multiply complex numbers, their magnitudes multiply and their phases add.

30. Proof for the third function (cont-d)

- Let us first prove that:
  - if the original instance has a solution ε_i,
  - then z is equal to the product of n values $z_i = 1 + \mathrm{i} \cdot \varepsilon_i \cdot t_i \in \mathbf{z}_i$.
- Indeed, since $|z_i| = \sqrt{1 + t_i^2}$, the product of the magnitudes is the desired value z.
- The angle α_i corresponding to each z_i is equal to $\alpha_i = \varepsilon_i \cdot \theta_i$.
- So the sum α of these angles is equal to $\sum\limits_{i=1} \varepsilon_i \cdot \theta_i$.
- Since $\theta_i = k \cdot s_i$, we conclude that $\alpha = k \cdot \sum\limits_{i=1}^{n} \varepsilon_i \cdot s_i$, i.e., $\alpha = 0$.
- So, this product $z_1 \cdot \ldots \cdot z_n$ has the right magnitude and the right angle and is, thus, equal to z.

31. Proof for the third function (cont-d)

- Vice versa, let us assume:
  - that z belongs to the range,
  - i.e., that z can be represented as the product $z_1 \cdot \ldots \cdot z_n$ for some $z_i \in \mathbf{z}_i$.
- In other words, for this product, the magnitude is equal to z, and the phase α is 0.
- For each value $z_i = 1 + \mathrm{i} \cdot y_i \in \mathbf{z}_i$, its magnitude is equal to $\sqrt{1 + y_i^2}$.
- Since $|y_i| \le t_i$, this magnitude cannot exceed $\sqrt{1 + t_i^2}$.
- The magnitude is equal to $\sqrt{1 + t_i^2}$ only for the two endpoints

$$y_i = \pm t_i.$$

32. Proof for the third function (cont-d)

- If for some i, we have $y_i \in (-t_i, t_i)$, then:
  - the resulting magnitude is the product of several numbers all of which are $\le \sqrt{1+t_i^2}$ and some are smaller;
  - thus, the magnitude of the product will be smaller than z.
- The magnitude of the product is equal to z.
- Thus, for each i, we have $z_i = 1 + \mathrm{i} \cdot \varepsilon_i \cdot t_i$ for some $\varepsilon_i \in \{-1, 1\}$.
- For each of these numbers z_i, the phase α_i is equal to $\varepsilon_i \cdot \theta_i$.
- The overall angle $\alpha = \sum\limits_{i=1}^{n} \alpha_i$ is equal to 0.
- So, we conclude that $\sum\limits_{i=1}^{n} \varepsilon \cdot \theta_i = 0$. Since $\theta_i = k \cdot s_i$, that $\sum\limits_{i=1}^{n} \varepsilon_i \cdot s_i = 0$.
- Hence, the original instance of the partition problem indeed has a solution.

33. Proof for the third function: general case

- In the general case, the values $\tan(\theta_i)$ and the product $\prod_{i=1}^{n} \sqrt{1+\tan^2(\theta_i)}$ are not rational.
- Then, we can:
  - as t_i, take rational values which are ε-close to t_i for some small $\varepsilon > 0$,
  - take $\mathbf{z}_i = 1 + \mathrm{i} \cdot [-t_i, t_i]$, and
  - take the interval $\mathbf{z} = [z_0 - \delta, z_0 + \delta]$ for some small δ, where z_0 is a rational number which is δ-close to the value $\prod_{i=1}^{n} \sqrt{1+\tan^2(\theta_i)}$.
- The proof then follows from the arguments presented for the second function.
- The proposition is proven.

34. First auxiliary result

- What if we can only measure the real part of the quantity y?
- In this case, checking the hypothesis means comparing the interval of all possible real values of $f(z_1, \ldots, z_n)$ with the given interval.
- Being able to solve this checking problem means being able to compute the range of real values and/or the range of imaginary values.
- It turns out that this problem is also NP-hard already for the product.
- Let $z = f(z_1, \ldots, z_n)$ be a complex-valued function.
- By a *problem of real-valued complex interval computations*, we mean the following problem;
  - given: a real-valued interval $\mathbf{x}$ and complex intervals $\mathbf{z}_1, \ldots, \mathbf{z}_n$,
  - check whether there exist values $z_i \in \mathbf{z}_i$ for which the real part of $f(z_1, \ldots, z_n)$ belongs to the interval $\mathbf{x}$.
- **First auxiliary result.** *For the product $f(z_1, \ldots, z_n) = z_1 \cdot \ldots \cdot z_n$, the problem of real-valued complex interval computations is NP-hard.*

35. Discussion

- In the real-valued case, for SUE expressions, we can feasibly compute the smallest interval containing the actual range.
- Namely, we can do it by using straightforward interval computations:
  - by replacing each operation with numbers in the original formula f
  - by the corresponding operation with intervals.
- For the product of complex numbers:
  - not only we do not get the smallest box at the end,
  - but the result of the corresponding operation-by-operation interval operations may actually depend on the order of multiplications
  - because for complex intervals, multiplication is, in general, not associative.

36. Discussion (cont-d)

- For example, for $(1-\mathrm{i})\cdot(1+\mathrm{i})\cdot([0,1]-\mathrm{i})$, we get:

$$(1+\mathrm{i})\cdot([0,1]-\mathrm{i}) = ([0,1]+1)+\mathrm{i}\cdot([0,1]-1) = [1,2]+\mathrm{i}\cdot[-1,0];$$

$$(1-\mathrm{i})\cdot([1,2]+\mathrm{i}\cdot[-1,0]) = ([1,2]+[-1,0])+\mathrm{i}\cdot([-1,0]-[1,2]) =$$
$$[0,2]+\mathrm{i}\cdot[-3,-1],$$

while $(1-\mathrm{i})\cdot(1+\mathrm{i})=2$ hence

$$2\cdot([0,1]-\mathrm{i}) = [0,2]-2\mathrm{i} \neq [0,2]+\mathrm{i}\cdot[-3,-1].$$

37. Proof of the first auxiliary result

- We will prove that computing the largest possible real part $\overline{x}$ of the product is NP-hard.
- In this proof, we use the same reduction as for the third function.
- In that proof, we showed that:
  - when $z_i \in \mathbf{z}_i$,
  - the product $z = z_1 \cdot \ldots \cdot z_n$ has a magnitude which cannot exceed $z \stackrel{\text{def}}{=} \prod_{i=1}^{n} \sqrt{1 + t_i^2}$.
- The real part of a complex number cannot exceed its magnitude.
- So, the largest possible value $\overline{x}$ of the real part cannot exceed z.
- The only possibility for $\overline{x}$ to be equal to z is when there is a point with real value z.
- Since the magnitude cannot exceed z, the imaginary part of this point must be 0.

38. Proof of the first auxiliary result (cont-d)

- Thus, the only way for $\overline{x}$ to be equal to z is to have z itself represented as a product of values $z_i \in \mathbf{z}_i$.
- We already know that checking this condition is NP-hard.
- Thus, computing $\overline{x}$ is also NP-hard.
- To take care of the general case,
  - when the tangents and the product are not all rational,
  - we can use of appropriate ε-close and δ-close rational numbers.

39. Second auxiliary result

- In some practical situations, we can directly measure the corresponding complex value z.
- In this case, as a result of the measurement, we get a rational-valued complex number $\widetilde{z}$ for which $|z-\widetilde{z}| \le \Delta$ for some known bound Δ.
- After this measurement, all we know about the actual (unknown) value z of this quantity is that this value belongs to the set

$$(\widetilde{z}, \Delta) \stackrel{\text{def}}{=} \{z : |z-\widetilde{z}| \le \Delta\}.$$

- This is known as *circular complex interval uncertainty.*
- Let $\widetilde{x}$, $\widetilde{y}$, and r be rational numbers.
- By a *circular complex interval*, we mean the set

$$\mathbf{z} = (\widetilde{z}, \Delta) \stackrel{\text{def}}{=} \{z : |z-\widetilde{z}| \le \Delta\}, \text{ where } \widetilde{z} = \widetilde{x} + \mathrm{i} \cdot \widetilde{y}.$$

40. Second auxiliary result (cont-d)

- By a *problem of circular complex interval computations* for a complex-valued function $z = f(z_1, \ldots, z_n)$, we mean the following problem;
  - given: circular complex intervals $\mathbf{z}$, $\mathbf{z}_1$, ..., $\mathbf{z}_n$,
  - check whether the circular complex interval $\mathbf{z}$ and the range $f(\mathbf{z}_1, \ldots, \mathbf{z}_n)$ have a common point.
- **Second auxiliary result.** *The problem of circular complex interval computations is NP-hard:*
  - *for the scalar (dot) product*
  - *and for the second moment.*

41. Proof of second auxiliary result

- Let us first consider the scalar (dot) product.
- The main idea is that similarly to the proof of the main result.
- Let us take $\mathbf{z}_i = \mathbf{t}_i = \sqrt{s_i} \cdot \left(1, \dfrac{\sqrt{2}}{2}\right)$ and $z = 0$.
- One can easily check that for complex numbers $z_i \in \mathbf{z}_i$, the phase takes the values from -45° to 45°.
- The phase is equal to 45° only at a point $\sqrt{s_i} \cdot (0.5 + \mathrm{i} \cdot 0.5)$.
- The phase is equal to -45° only at a point $\sqrt{s_i} \cdot (0.5 - \mathrm{i} \cdot 0.5)$.
- When we multiply complex numbers, their phases add up.
- Thus, for the product $z_i \cdot t_i$, the angle is always between -90° and 90°.
- Hence, the real part of the product is always non-negative.

42. Proof of second auxiliary result (cont-d)

- So, the real part of the sum $\sum\limits_{i=1}^{n} z_i \cdot t_i$ is also always non-negative.
- The only possibility for this real part to be 0 is:
  - when the real parts of all the terms in the sum are equal to 0,
  - i.e., when for each i, the phase of the product $z_i \cdot t_i$ is equal to either 90° or to -90°.
- This, in turn, is possible only if:
  - either both z_i and t_i have phases 45°
  - or both z_i and t_i have phases -45°.
- In the first case, we have $z_i = t_i = \sqrt{s_i} \cdot (0.5 + \mathrm{i} \cdot 0.5)$, hence
$$z_i \cdot t_i = 0.5 \cdot s_i \cdot \mathrm{i}.$$
- In the second case, we have $z_i = t_i = \sqrt{s_i} \cdot (0.5 - \mathrm{i} \cdot 0.5)$, hence
$$z_i \cdot t_i = -0.5 \cdot s_i \cdot \mathrm{i}.$$

43. Proof of second auxiliary result (cont-d)

- In both cases, we have $z_i \cdot t_i = 0.5 \cdot \varepsilon_i \cdot s_i \cdot \mathrm{i}$ for some $\varepsilon_i \in \{-1, 1\}$.
- Thus, the imaginary part of the sum $\sum\limits_{i=1}^{n} z_i \cdot t_i$ is equal to $0.5 \cdot \sum\limits_{i=1}^{n} \varepsilon_i \cdot s_i$.
- This imaginary part is equal to 0:
  - if and only if $\sum\limits_{i=1}^{n} \varepsilon_i \cdot s_i = 0$,
  - i.e., if and only if the original instance of the partition problem has a solution.
- To take care of the fact that the square roots are, in general, not rational, we can use appropriate ε-close and δ-close rational numbers.
- The proof for the second moment is similar.

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