

Causality: Hypergraphs, Matter of Degree, Foundations of Cosmology

Cliff Joslyn¹, Andres Ortiz-Muñoz², Edgar Daniel Rodriguez Velasquez³,
Olga Kosheleva⁴, and Vladik Kreinovich⁵

¹Mathematics of Data Science, Pacific Northwest National Laboratory,
Seattle, Washington 98109, USA, cliff.joslyn@pnl.gov

²Santa Fe Institute, Santa Fe, New Mexico 87501, USA, aortiz@santafe.edu

³⁻⁵Departments of ³Civil Engineering, ⁴Teacher Education, and ⁵Computer Science
University of Texas at El Paso, El Paso, Texas 79968, USA
edrodriguezvelasquez@miners.utep.edu, olgak@utep.edu, vladik@utep.edu

1. Formulation of the problem

- Causality is an important notion in many areas of research, in many practical situations.
- To be able to solve the corresponding practical problems, it is important to have an adequate description of what is causality.
- Because this notion is so important, there are many formal descriptions of causality in physics and in many other areas.
- The very fact that there are many different description is a good indication that the current definitions are not perfect.
- And most authors writing about causality explicitly state that their definitions:
 - while useful in many practical cases,
 - do not fully capture the intuitive idea of causality.

2. Formulation of the problem (cont-d)

- One of the important aspects of causality
 - that is not properly captured by the current definitions
 - is that in some situations, causality is a matter of degree.
- This has been stated, on many occasions, by Lotfi Zadeh and other researchers working in fuzzy techniques.
- In this paper, we show:
 - how we can extend the current descriptions of causality
 - to take this – and other – important aspects of causality into account.

3. Two main areas where the notion of causality is used

- In order to motivate the following definitions, let us recall what we usually mean by causality.
- This notion is very important both:
 - in everyday life – and, as an extension, in Artificial Intelligence (AI), and
 - in physics.
- Let us briefly describe how this notion is used in both application areas.

4. Two main areas where causality is used (cont-d)

- We will then emphasize the fact that is not as widely known as it should be – that:
 - while in these two areas, we use the same term “causality”,
 - in these two areas, we talk about two *different* aspects of this notion.
- Thus, a proper formalization of this notion must take this difference into account.

5. Commonsense and AI aspect of causality: an example

- If a road pavement needs repairs while it was supposed to still last, we need to know what caused it.
- It could be that there were mistakes in design.
- It could be that the design was correct, but:
 - the wrong materials were used,
 - or, more generally, that some mistakes were made when this road segment was built.
- It could be that:
 - there were unusual extreme weather conditions that were not predicted
 - and that caused the premature pavement deterioration.
- It could be that the load on this road segment was much higher than anticipated, and this caused the pavement deterioration.

6. Commonsense and AI aspect of causality (cont-d)

- It could be that the formulas used to describe how pavement deteriorates with time were incorrect, etc.
- In this sense, when we say that a caused b , this means that:
 - if – everything else being equal – we would not have a ,
 - then we would not see b either.
- For example:
 - if we conclude that the use of a wrong asphalt mix caused the pavement deterioration,
 - this means that if proper material was used, the road would not have deteriorated, and
 - that in a similar situation, if we use the proper material, the road will not deteriorate before its expected lifetime.

7. Causality in physics: an example

- The notion of causality is very important in physics too, but there, it means something else.
- Specifically, in physics:
 - when we say that an event e causally affect an event e' ,
 - this means that a change in e *can* make some change in e' .

8. What is the difference between these two aspects

- Suppose that a soccer player kicks a ball and it goes into the opponent's gate.
- Then, both from the commonsense viewpoint and from the physics viewpoint, this kick was the cause of the resulting goal.
- So far so good, it may look like these two notions are the same.
- However, here is a counterexample.
- In real life, there is usually some wind, and that wind slightly affects the trajectory of the ball.
- From the viewpoint of common sense, the wind was clearly *not* the cause of the goal.
- Unless it was such a strong gust of wind that the ball – that was originally supposed to miss – was blown into the gate.

9. What is the difference between these two aspects (cont-d)

- However, from the viewpoint of theoretical physics:
 - the wind *did* have a causal effect on the ball's trajectory
 - in contrast, e.g., to the wind 1000 km away at this same moment that did not have any way to causally affect the ball's trajectory.
- In short, in the first approximation, we have the following:
- In common sense, a causes b if:
 - whenever we have a , we have b as well,
 - i.e., the presence of a determines the presence of b .
- In theoretical physics, a causally affects b :
 - if a has *some* effect on b ,
 - but does not necessarily uniquely determines *all* the aspects of b .

10. Towards a more realistic description

- The above first-approximation description implicitly assumed that everything is deterministic.
- In real life – and in physics – we can rarely fully predict the future events.
- At best, we can predict the *probabilities* of different future events.
- With this in mind, the proper description of the two aspects of causality takes the following form:
- In common sense:
 - we say that a causes b
 - if a *uniquely determines* the probability distribution of possible values of the quantities characterizing the event b .

11. Towards a more realistic description (cont-d)

- In theoretical physics:
 - we say that a causally effects b
 - if a has *some effect* on the probability distribution of possible values of the quantities characterizing the event b .
- Of course, these explanations are still approximate.
- In reality, in common sense:
 - we take into account that some minor changes may still happen,
 - so we can have slightly different probability distributions of b .
- However, from the practical viewpoint, this description provides a good description of real-life situations.

12. How causality is usually described

- The usual description of causality is based on the following two natural properties of causality.
- If a causally influences b , then b cannot casually influence a .
- Unless, of course, a and b are the same – and we extend the definition of causality to this trivial case.
- If a causally influences b and b causally influences c , then a casually influences c .
- It makes sense, by the way, to formally extend the notion “ a causally influences b ” to the case when $a = b$.
- Indeed, as we have mentioned earlier, this notion means that a change in a can cause a change in b .
- And, of course, a change in a causes a change in a .
- The same two properties are true if instead of “causally influences”, we use “causes”.

13. How causality is usually described (cont-d)

- Let us denote “ a causally influences b ” or “ a causes b ” by $a \leq b$.
- Let us extend the definition to allow the event to casually influence itself.
- Then, we get the following properties for all a , b , and c :
 - $a \leq a$
 - if $a \leq b$ and $b \leq a$, then $a = b$; and
 - if $a \leq b$ and $b \leq c$, then $a \leq c$ (*transitivity*).
- In mathematics, a binary relation with these properties is known as an *order relation*.
- It is sometimes called *partial order*.
- This is done to distinguish it from the case of *linear (total) order*, when for every a and b , we have either $a \leq b$ or $b \leq a$.
- So, causality is usually described as a *partial order*.

14. The first (known) limitation of the usual description

- In practice, sometimes, several phenomena have to be present to cause the observed effect.
- For example:
 - a building may be designed so as to withstand an earthquake, and it can be designed to withstand a tsunami,
 - but if both unexpectedly happen at about the same time – as happened in Japan in 2011 – the building may collapse.
- In this case, the cause of the collapse:
 - is not just the earthquake – an earthquake itself would not have caused the collapse,
 - and not the tsunami,
 - but the fact that both were present.

15. The first limitation of the usual description (cont-d)

- A more mundane example is that:
 - to open a bank vault (or even to open a safe deposit box stored in a bank), you need two keys,
 - none of them would open the vault by itself.
- It is difficult to describe this phenomenon by an order relation between the events.
- A similar phenomenon can be imagined in physics, where it is important to describe cases in which:
 - an event a cannot affect an event e'
 - unless some other event e'' was present.

16. The second (known) limitation of the usual description

- The second known limitation of the current definition of causality as a partial order is that:
 - it does not take into account
 - that causality is a matter of degree.
- In some cases, a strongly influences b , in other cases, it only affects b with some degree.
- As a result:
 - when we have a long causal chain in which a_1 affects a_2 , a_2 affects a_3 , etc., until a_{n-1} affects a_n ,
 - we can formally say that a_1 affects a_n .
- This is usually expressed by saying that the flapping of a butterfly's wings affects a future storm on another continent.
- However, in reality, the degree of this effect decreases at every link of this chain.

17. The second limitation of the usual description (cont-d)

- So, at the very end, the effect is not practically noticeable.
- What we do in this paper is show how to take these two features of causality into account.

18. How to take into account that we may need several events to cause the observed phenomenon

- In the deterministic approximation to the traditional commonsense description, the relation $a \leq b$ means that:
 - if we know the state of a ,
 - then we can uniquely predict the state of b .
- In the case that we want to describe, knowing the state of a is not enough to uniquely predict the state of b .
- To make this prediction, we need to know the state of several events a_1, \dots, a_n .
- In other words, here:
 - instead of knowing the state of a single event a ,
 - we need to know the state of all the events from some set A .

19. How to take into account that we may need several events to cause the observed phenomenon (cont-d)

- It is therefore reasonable to consider the corresponding relation $A \leq b$ that means that:
 - once we know the states of all the events from the set A ,
 - then we can uniquely determine the state of the event b .
- Of course, the way we described it means that:
 - if we add other elements to the set A ,
 - we would still get this relation $A \leq b$.
- We need to distinguish:
 - “true causality” – which we will denote by $A < b$ –
 - from the above relation $A \leq b$.
- A natural way to define $A < b$ is b is not an element of the set A and no proper subset of A influences the event b .

20. How to take into account that we may need several events to cause the observed phenomenon (cont-d)

- So, if $A < b$, then we cannot have $A' < b$ for any proper subset $A' \subset A$, $A' \neq A$.
- In particular, when A is the empty set, we can have $\emptyset < b$.
- This means that all the properties of the event b are pre-determined, nothing can affect them.

21. What is the natural analogue of avoiding the causality loop, when a causes b and b causes a

- The original definition of causality avoids non-intuitive loops by requiring that if $a \leq b$ and $b \leq a$, then $a = b$.
- In our case, a proper description is that if $A < b$, then we cannot have $B < a$ for any element $a \in A$ and for any set B including b .

22. What is the natural analogue of transitivity

- In the traditional transitivity, if b causes c , and a causes b , then a causes c .
- What if, in our case, we have $B < c$?
- Then, to be able to conclude that $A < c$:
 - we need to make sure that all events $b \in B$ can be predicted by A ,
 - i.e., that $A < b$ for all $b \in B$.
- In this case, the set A uniquely determine c .
- However, maybe not all elements of A are needed to predict c .
- So, all we can conclude is that $A \leq c$, i.e., that some subset of the set A causes c .

23. Resulting definition

- Let U be a set.
- By a *causality relation* on U , we mean a relation $<\subseteq 2^U \times U$ between subsets of U and elements of U s.t. the following properties hold:
 - if $A < b$, then $b \notin A$;
 - if $A < b$, $A' \subset A$ and $A' \neq A$, then $A' \not< b$;
 - if $A < b$, then for any $a \in A$ and for any set B containing b , we cannot have $B < a$;
 - if $A < b$ for all $b \in B$ and $B < c$, then we have $A' < c$ for some subset $A' \subseteq A$ of the set A .

24. Terminological comment

- Partial order can be described by a directed acyclic graph (DAG), in which:
 - objects are vertices, and
 - pairs (a, b) for which $a \leq b$ and $a \neq b$ are edges.
- In this section, we consider a generalization of this notion to the case when the relation $A \leq b$ involves, in general:
 - more than one element $a \in A$ and thus,
 - more than two objects.
- Such a generalization of a graph – in which we consider subsets consisting of more than 2 elements – is known as a *hypergraph*.
- From this viewpoint, the above definition provides a generalization of the notion of a directed acyclic graph to hypergraphs.

25. An alternative description

- So far, we have dealt with the “determination” relation $A \leq b$ meaning that knowing A uniquely determines b .
- In terms of the causality relation $A < b$, this means that either $b \in A$ or there exists a subset $A' \subset A$ for which $A' < b$.
- A somewhat inconvenient issue with this notion is that:
 - in contrast to the original definition of causality $a \leq b$ that relates two objects of the same type,
 - the relation $A \leq b$ related objects of different types: sets and elements.
- It is therefore natural to extend this relation to relation between two objects of the same type, namely, between sets.
- We can define $A \leq B$ if knowing all information about A provides us with the ability to predict all the values of all the elements $b \in B$.

26. An alternative description (cont-d)

- This possibility means that we can predict the values of each state $b \in B$, i.e., that we have $A \leq b$ for all $b \in B$.
- This new relation between sets has the following natural properties:
 - if $B \subseteq A$, then $A \leq B$;
 - if $A \leq B$ and $A \subseteq A'$, then $A' \leq B$;
 - if $A \leq B$ and $B' \subseteq B$, then $A \leq B'$.
- In terms of the new relation, transitivity takes a very straightforward form: if $A \leq B$ and $B \leq C$, then $A \leq C$.
- Other properties of the causality relation can also be described in terms of this relation $A \leq B$ if we take into account:
 - that $A \leq b$ simply means $A \leq \{b\}$, and
 - that, as we have mentioned, $A < b$ means that $b \notin A$ and that for all proper subsets $A' \subset A$, we have $A' \not\leq b$.

27. How to take into account that causality is a matter of degree

- Another limitation of the traditional approach to causality is that this approach is black-and-white (crisp).
- In this approach, either an event a can influence event b or it cannot.
- In practice, at each moment of time, due to imperfection of measuring instruments, we can only detect effects of certain size.
- So, we can only confirm for certain pairs (a, b) that a can influence b .
- As technology progresses, we can detect weaker and weaker effects – and thus, confirm that other pairs are also causally connected.
- A natural way to describe this situation is to assign, to each pair of events (a, b) , a degree $d(a, b)$ to which a can influence b .
- The larger the effect, the larger this degree.
- At each moment of time, we can measure the effect with a certain accuracy.

28. How to take into account that causality is a matter of degree (cont-d)

- Thus, we know this degree with the similar accuracy.
- A natural way to describe such uncertainty is by saying that:
 - we know the approximate value \tilde{d} of this degree, and
 - we know the accuracy δ .
- So, the only thing we know about the actual (unknown) degree d is that this degree is located somewhere in the interval $[\tilde{d} - \delta, \tilde{d} + \delta]$.

29. What are the consequences of taking this into account

- When we consider causality as crisp, then transitivity can be used indefinitely.
- If a_1 causes a_2 , and a_2 causes a_3 , etc., and a_{n-1} causes a_n , then a_1 causes a_n .
- In practice, the effect decreases with each transition:
 - maybe a_1 has a strong effect on a_2 ,
 - maybe a_2 also has a strong direct effect on a_3 ,
 - but the resulting indirect effect of a_1 on a_3 is smaller.
- Indeed, every time we have an effect, there is also some random influence that interferes with this effect.
- After many steps, the original effect becomes practically non-observable.

30. What are the consequences of taking this into account (cont-d)

- Similarly:
 - if we transmit a message many times through noisy channels,
 - the message will become corrupted more and more and eventually, will become completely unrecognizable.
- In precise terms, instead of crisp transitivity, it is natural to expect the degree-based (fuzzy) version.
- In this version:
 - for each a , b , and c ,
 - the degree $d(a, c)$ to which a affects c is bounded by an “and”-operation (t-norm) $f_{\&}$ applied to the degrees $d(a, b)$ and $d(b, c)$:

$$d(a, c) \geq f_{\&}(d(a, b), d(b, c)).$$

- This leads to a more realistic picture of causality.

31. What are the consequences of taking this into account (cont-d)

- For example, we can use a t-norm for which we can have $f_{\&}(a, b) = 0$ for some $a > 0$ and $b > 0$.
- Then eventually, the degree of influence can get to 0.
- Thus, we can have, for example:
 - a closed time loop, in which going to the future brings us back to the past,
 - provided that the path was sufficiently long.
- This possibility makes physical sense: in billions of years, all traces of the original event will disappear.

32. Interesting consequence of this idea

- The idea of causality as a matter of degree does not just provide us with a better fit with physical intuition.
- In this section, we will show that this idea also helps us to understand a certain fact from fundamental physics.
- Namely, it helps understand a somewhat counterintuitive transition between:
 - special relativity theory (and local effects of generic relativity) – that describe the local space-time, and
 - cosmology that describes the global structure of space-time.
- Let us explain what is counterintuitive in this transition.
- One of the main principles of special relativity theory is the relativity principle.

33. Interesting consequence of this idea (cont-d)

- According to this principle:
 - by observations inside the system,
 - there is no way to determine whether the system is at rest or moving with a constant speed in the same direction.
- This principle was first formulated by Galileo who noticed that:
 - when a ship is moving in a calm sea,
 - then, if we are inside a cabin with no windows, we cannot tell whether the ship is moving or not.
- Einstein combined this principle with the empirical fact that the speed of light c_0 has the same value for all observers.
- He concluded that a priori, there is no fixed time coordinate.
- Time (and corresponding notion of simultaneity) differ for different observers.

34. Interesting consequence of this idea (cont-d)

- Neither time interval Δt nor spatial distance ρ are invariant: they change from observer to observer,
- The only quantity that remains invariant is the so-called *proper time*, which, in distance units, takes the form $\tau = \sqrt{c_0^2 \cdot (\Delta t)^2 - \rho^2}$.
- The impossibility to separate time and space is the main reason why in special relativity:
 - we talk about 4-dimensional space-time, and
 - its division into space and time depends on the observer.
- Transformation between coordinate systems corresponding to different observers is described by so-called Lorentz transformations.
- In contrast, in most cosmological models:
 - there is a very clearly determined time coordinate,
 - there is a clear separation into space and time.

35. Interesting consequence of this idea (cont-d)

- Why is that? Why cannot we have cosmology that has the same symmetries as local space-time?
- Let us first consider the usual – crisp – causality.
- Then, the only answer to the above question is that observations:
 - support cosmological models in which time and space are separated, and
 - support equations of General Relativity that lead to these models.
- In this description, there is nothing fundamental about these types of cosmologies.
- From the purely mathematical viewpoint, there is nothing wrong with considering space-time of special relativity.
- Interestingly, the situation changes if we take into account that causality is a matter of degree.

36. Interesting consequence of this idea (cont-d)

- It turns out that:
 - if we make this assumption,
 - then we cannot keep all the symmetries of special relativity, so
 - cosmological split between time and space becomes inevitable.
- Let us describe this result in precise terms.
- To come up with this explanation, let us recall that in Special Relativity, the speed of light c is the largest possible speed.

37. Interesting consequence of this idea (cont-d)

- So:

- an event $e = (t, x_1, x_2, x_3)$ can influence the event

$$e' = (t', x'_1, x'_2, x'_3)$$

- if and only if we can get from e to e' by traveling with a speed not exceeding the speed of light.

- In other words, e can influence e' if and only if:

- the distance between the corresponding spatial points (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) is smaller than or equal to

- the distance $c_0 \cdot (t' - t)$ that someone traveling with the speed of light can cover during the time interval $t' - t$.

38. Crisp causality: formal definition

- Let c be a positive constant; we will call it *speed of light*.
- By *space-time of special relativity*, we mean an ordered set (M, \leq) in which:
 - the set $M = \mathbb{R}^4$ is the set of all 4-D tuples $e = (t, x_1, x_2, x_3)$ of real numbers, and
 - the ordering relation has the form

$$e = (t, x_1, x_2, x_3) \leq e' = (t', x'_1, x'_2, x'_3) \Leftrightarrow$$

$$c_0 \cdot (t' - t) \geq \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}.$$

- If $e = (t, x_1, x_2, x_3) \leq e' = (t', x'_1, x'_2, x'_3)$, then by the *proper time* $\tau(e, e')$ we mean the quantity

$$\tau(e, e') = \sqrt{c_0^2 \cdot (t' - t)^2 - (x_1 - x'_1)^2 - (x_2 - x'_2)^2 - (x_3 - x'_3)^2}.$$

39. Crisp causality: formal definition (cont-d)

- We say that a mapping $f : M \rightarrow M$ is a *symmetry*:
 - if it preserves causality relation and preserves proper time,
 - i.e., if $e \leq e'$ is equivalent to $f(e) \leq f(e')$ and $\tau(e, e') = \tau(f(e), f(e'))$ for all events e and e' for which $e \leq e'$.
- It is known that:
 - for every two pairs (e, e') and (g, g') for which $\tau(e, e') = \tau(g, g')$,
 - there exists a symmetry f that transforms e into g and e' into g' .

40. Causality as a matter of degree: towards a precise definition

- Let us now describe what we mean by taking into account that causality is a matter of degree.
- We will call the corresponding descriptions *realistic causality functions*.
- Their definition uses the notion of an “and”-operation, so let us first define this auxiliary notion.
- By an “*and*”-operation, we mean a continuous function $f_{\&} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ for which $f_{\&}(1, 1) = 1$.
- Every t-norm – as defined in fuzzy logic – satisfies this condition.
- However, this definition is much weaker than the usual definition of t-norm in fuzzy logic.
- It allows many functions which are not t-norms – e.g., they are not necessarily associative.

41. Causality as a matter of degree: towards a precise definition (cont-d)

- The reason for us formulating such a weak definition is that we want to prove the main result of this section under most general assumptions.
- For example, our result stands if we consider non-commutative and non-associative operations instead of t-norms.
- Continuity at the point $(1, 1)$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \geq 1 - \delta$ and $y \geq 1 - \delta$, then $f(x, y) \geq 1 - \varepsilon$.
- We want to define a function $d(e, e')$ that describes:
 - for each two events e and e' for which $e \leq e'$,
 - the degree to which e can influence e' .
- What are the natural property of such a function?

42. Causality as a matter of degree: towards a precise definition (cont-d)

- Of course, it is a fact:
 - that each event e “causes” itself,
 - in the sense that if we know the event e , we can uniquely describe this same event.
- So, we must have $d(e, e) = 1$.
- The fact that causality is a matter of degree means that the case when $e = e'$ should be the only case when we have $d(e, e') = 1$.
- In all other cases, we must have $d(e, e') < 1$.
- In physics, most dependencies are continuous, so it is reasonable to require that the function $d(e, e')$ should also be continuous.
- Finally, for the case when $e \leq e' \leq e''$, we should have

$$d(e, e'') \geq f_{\&}(d(e, e'), d(e', e'')).$$

Thus, we arrive at the following definition.

43. Causality as a matter of degree: precise definition

- Let $f_{\&}$ be an “and”-operation.
- By a *realistic causality function* on (M, \leq) , we mean a continuous function $d(e, e')$:
 - with values from the interval $[0, 1]$
 - that is defined for all pairs (e, e') for which $e \leq e'$ and
 - that satisfies the following two properties:
 - we have $d(e, e') = 1$ if and only if $e = e'$, and
 - we have $d(e, e'') \geq f_{\&}(d(e, e'), d(e', e''))$ for all $e, e', e'' \in M$.
- We say that a realistic causality function is *invariant* with respect to a symmetry f if $d(e, e') = d(f(e), f(e'))$ for all $e \leq e'$.

44. Main result

- **Result.** *No realistic causality function is invariant with respect to all the symmetries.*
- This result explains that:
 - when we take into account that causality is a matter of degree,
 - then we cannot have a space-time that has the same invariance properties as the space-time of Special Relativity.
- Some corresponding symmetries have to be abandoned.
- This explains why none of the current cosmological models has all these symmetries.

45. Proof

- Let us prove our statement by contradiction.
- Let us assume that a realistic causality function d is invariant with respect to all the symmetries.
- Let us denote, for all $t \geq 0$, $e(t) \stackrel{\text{def}}{=} (t, 0, 0, 0)$.
- We want to prove that in this case, we will have $d(e(0), e(1)) = 1$.
- This will contradict to the first part of the definition of the realistic causality function.
- According to this part, $d(e, e') = 1$ is only possible when $e = e'$.
- To prove the equality $d(e(0), e(1)) = 1$, we will prove that for every $\varepsilon > 0$, we have $d(e(0), e(1)) \geq 1 - \varepsilon$.

46. Proof (cont-d)

- According to the second comment after the definition of an “and”-operation, for each $\varepsilon > 0$, there exists a number $\delta > 0$ for which:
 - if $x \geq 1 - \delta$ and $y \geq 1 - \delta$,
 - then $f_{\&}(x, y) \geq 1 - \varepsilon$.
- Thus:
 - if we have an event e for which $d(e(0), e) \geq 1 - \delta$ and $d(e, e(1)) \geq \delta$,
 - then we have $f_{\&}(d(e(0), e), d(e, e(1))) \geq 1 - \varepsilon$
 - and hence, since $d(e(0), e(1)) \geq f_{\&}(d(e(0), e), d(e, e(1)))$, we have $d(e(0), e(1)) \geq 1 - \varepsilon$.
- To find such e , let us recall that, as we have mentioned:
 - every two pairs (e, e') and (g, g') for which the proper time is the same
 - can be transformed into each other by an appropriate symmetry f : $f(e) = g$ and $f(e') = g'$.

47. Proof (cont-d)

- The realistic causality function is invariant with respect to all the symmetries.
- This means that $d(e, e') = d(f(e), f(e')) = d(g, g')$.
- In other words:
 - if two pairs have the same proper time,
 - then these pairs have the same degree of causality.
- In mathematical terms, this means that the realistic causality function is a function of proper time, i.e., that

$$d(e, e') = F(\tau(e, e')) \text{ for some function } F(x).$$

- In particular, for $e(t) = (t, 0, 0, 0)$ with $t \geq 0$, we have

$$\tau(e(0), e(t)) = c_0 \cdot t.$$

- So, for these pairs, we have $d(e(0), e(t)) = F(c_0 \cdot t)$.

48. Proof (cont-d)

- The function d is continuous and $e(t) \rightarrow e(0)$ as $t \rightarrow 0$.
- In the limit $t = 0$, we get $d(e(0), e(0)) = 1$.
- Thus, the function $F(x)$ is also continuous, tending to $F(0) = 1$ as $x \rightarrow 0$.
- By definition of the limit, this means that for every $\delta > 0$, there exists an $\nu > 0$ such that:
 - if $x \leq \nu$,
 - then $F(x) \geq 1 - \delta$.
- Let us take $e = (t/2, c_0 \cdot t/2 - \alpha, 0, 0)$.
- Here, as one can easily check, $e(0) \leq e \leq e(1)$, $\tau(e(0), e) = \tau(e, e(1))$ and the common value of proper time tends to 0 as $\alpha \rightarrow 0$.
- Thus, for sufficiently small α , we have $\tau(e(0), e) \leq \nu$.

49. Proof (cont-d)

- Hence, we have $d(e(0), e) = F(\tau(e, e(0))) \geq 1 - \delta$ and similarly

$$d(e, e(1)) \geq 1 - \delta.$$

- We have already shown that these two inequalities imply that

$$d(e(0), e(1)) \geq 1 - \varepsilon.$$

- Since this is true for every $\varepsilon > 0$, this means that $d(e(0), e(1)) = 1$.
- This contradicts to the definition of the realistic causality function.
- This contradiction proves that our assumption was wrong.
- Thus, indeed, no realistic causality function can be invariant with respect to all the symmetries.

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