From Historically First "Unary" Numbers, Through Egyptian Fractions, Roman Numerals, Leibniz's Binary Numbers and Kepler's Fractions to Modern Ideas Such as Calkin-Wilf Tree: A Unified Approach to Representing Natural Numbers and Fractions

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1. Introduction

- In elementary mathematics classes, students are often overwhelmed by different representations of numbers and corresponding operations:
  - usual fractions,
  - decimal representations,
  - binary numbers, etc.
- What often helps is when students:
  - learn the history of these representations,
  - learn the limitations of seemingly reasonable representations like Roman numerals, and
  - learn how other representations overcame these limitations.

2. Introduction (cont-d)

- Still, history was developed somewhat randomly, so the historical sequence is still somewhat chaotic.
- We believe that providing a systematic unified approach for all these representations would:
  - help describe their sequence in a more logical way and thus,
  - help the students even more.

3. Purpose and objectives of the study

- The main purpose of this study is to provide a systematic unified approach to historical number representations.
- We believe that exposing students (and teachers) to such a unified approach:
  - will help them better understand the relation between different number representations, and
  - thus, will make it easier for them to master the corresponding techniques.

4. Literature review

- There exist many books and paper on historical number systems and number representations.
- These books and papers:
  - describe different representation of numbers,
  - describe the practical motivation for these representations,
  - describe how these representations historically evolved, and
  - what were their strengths and limitations.
- The problem is that most of these papers concentrated:
  - on a single historical representation,
  - or, at best, on a few related historical representations.
- These papers do not provide a systematic unified view of different representations.
- Thus, they make it difficult to use several different representations when teaching students.

5. Methodology

- In our analysis, we explore the relation between:
  - the foundations of arithmetic, especially related definitions of numbers, and
  - the resulting representations.

6. Analysis of the problem

- A set of objects is usually described by:
  - listing basic objects and
  - listing allowed operations, operations that enable us to construct new objects.
- For numbers, this means that:
  - we list several basic numbers, and
  - we list operations that can be used to construct new numbers.
- Let us describe what types of numbers and what operations we will consider in this talk.
- In this talk, we will focus of two types of numbers:
  - non-negative integers, also known as natural numbers $0, 1, 2, \ldots,$ and
  - non-negative fractions, also known as non-negative rational numbers.

7. What operations are possible?

- For integers, natural arithmetic operations are addition, subtraction, and multiplication.
- It is also natural to consider powers, since a power is nothing else but a repeated multiplication.
- For rational numbers, there is an additional arithmetic operation: division.
- Addition, subtraction, and multiplication are exactly operations which are directly hardware supported in modern computers.
- To be more precise:
  - only addition and multiplication are directly hardware supported,
  - subtraction $a-b$ is reduced to addition $a+(-b)$ by using so-called 2's-complement representation of negative numbers.
- All other computations are performed, in effect, as a sequence of additions and multiplications.

8. What operations are possible (cont-d)

- For example:
  - when we ask a computer to compute the value of a special function like $\exp(x)$ or $\sin(x)$,
  - the computer actually computes the sum of the first few terms of the corresponding Taylor series,
  - i.e., computes the value of the corresponding polynomial by performing the appropriate sequence of additions and multiplications.
- How is division represented in a computer?
- Since computers use binary numbers, the simplest division for computers is division by 2.
- It simply shifts the binary point, similarly to how division by 10 is the easiest for us humans, who use decimal system.
- A general division a/b is implemented in a computer as the product

$$a \cdot (1/b).$$

9. What operations are possible (cont-d)

- Here, the inverse $1/b$ is actually computed as a sequence of additions and multiplications.
- Also, in the computers:
  - simplest special cases of addition and subtraction – namely, the operations of adding one and subtracting one,
  - are usually hardware supported as separate operations;
  - in computer science, these operations are usually denoted by $++$ (next integer) and $--$ (previous integer).
- These separate operations make sense.
- Indeed, adding one is a very frequent operation corresponding to counting.
- Adding one is much faster than the general addition.
- Thus, implementing adding 1 as a separate hardware supported operation enables us to decrease computation time.

10. What number systems do we get?

- Let us analyze what number systems we can get if we use some of these operations.
- We will start with number systems describing natural numbers.
- Then, we will move to number systems describing non-negative fractions.
- It is important to notice that:
  - while this way, we do describe, in a systematic way, practically all historically proposed number systems,
  - the resulting systematic order in which we present these systems will be sometimes different from the historic order,
  - i.e., the order in which these systems are usually represented in books on history of mathematics.

11. What if we only use ++: from historically first representation of numbers to Peano arithmetic

- Let us start with the case when:
  - to construct new natural numbers,
  - we use only one arithmetic operation, namely the simplest arithmetic operation ++ of adding 1.
- What basic natural numbers do we need to have in such a description?
- We want to represent 0, and 0 cannot be described as a natural number + 1.
- We therefore have to have 0 as one of the basic numbers.
- There is no need to have any other basic number.
- Indeed, once we have 0, we can construct any natural number by adding an appropriate number of 1s:

$$1 \text{ is } 0 + 1,\ 2 \text{ is } 0 + 1 + 1,\ \text{etc.}$$

12. What if we only use ++: from historically first representation of numbers to Peano arithmetic (cont-d)

- This is exactly how natural numbers are described in so-called Peano arithmetic.
- Peano arithmetic is the most well-known and the most widely used formal description of natural numbers:
  - 0 is a natural number, and
  - if n is a natural number, then the next value – which in Peano arithmetic is usually denoted by $s(n)$ – is also a natural number.
- What representation of natural numbers do we get this way?
- In this approach, each natural number can be described by listing all adding-one operations that we need to apply to 0 to get this number:
  - if we denote one application of the adding-one operation by a vertical line,
  - then numbers 1, 2, 3, 4, etc. are represented as, correspondingly, I, II, III, IIII, etc.

13. What if we only use ++: from historically first representation of numbers to Peano arithmetic (cont-d)

- This is known as the unary number system, since
  - we use only one symbol to represent all natural numbers,
  - as opposed to, e.g., the usual decimal system where we use ten different symbols 0, 1, ..., 9.
- The unary system is exactly how our ancestors counted.
- For example, if they needed to record that someone borrowed 13 sheep, they scratched 13 lines on a stick.
- For small numbers, this is how 1, 2, 3, and sometimes 4 were represented in Roman numerals.
- This is how such numbers are marked on grandfather's clocks.

14. What if we only use ++: from historically first representation of numbers to Peano arithmetic (cont-d)

- An important feature is that in this representation:
  - there is exactly one way to represent a given natural number,
  - namely, to represent a number n, we write down II...I, where I is repeated n times.

15. What if we also use addition: Biblical numbers

- Let us recall that:
  - the simplest arithmetic operation is addition, and
  - the ++ operation of adding one is the simplest particular case of this operation.
- In the previous section, we considered the case when we only use ++.
- A limitation of this representation is that it takes too much space: e.g., to represent number 1000, we need to use 1000 symbols.
- To make representations shorter, a natural idea is to use other arithmetic operations.
- The natural next operation is full addition; so:
  - instead of a single symbol, we use several symbols describing several different natural numbers, and
  - we form other natural numbers by adding the given numbers.

16. What if we also use addition: Biblical numbers (cont-d)

- Such systems have indeed been ubiquitous in the ancient world.
- For example, to describe numbers in the Hebrew Bible:
  - the first 9 letters of the Hebrew letters of the Hebrew alphabet were assigned values 1 through 9,
  - the next 9 letters 10, 20, . . . , 90,
  - then 100, etc.
- This provides for a more compact representation.
- For example, we only need 3 symbols to represent the number 144: the symbols for 100, 40, and 4.
- The disadvantage is that, in principle, the representation stops being unique.
- For example, the number 16 can be, in principle, represented both as $10 + 6$, and as $9 + 7$.
- Such a 9-using representation was actually used.

17. What if we also allow subtraction: Roman numerals

- To get an even more compact representation, it is reasonable to allow other operations.
- The simplest operation that we did not use was subtraction.
- This is how the Roman numerals were built.
- We start with symbols describing several natural numbers:

$$I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, \text{ and } M = 1000.$$

- Then we use addition and subtraction to represent other natural numbers:
  - if a symbol for a smaller number follow the symbol for a larger number, it means addition,
  - otherwise it means subtraction.
- For example, 144 was represented as CXLIV, which means

$$100 + (50 - 10) + (5 - 1).$$

18. What if we allow subtraction: Roman numerals (cont-d)

- This system also allows several different representations for the same number.
- For example, the number 4 was sometimes represented as IIII and sometimes as IV.

19. What if we allow the simplest possible multiplication – multiplication by 2: binary numbers

- The next-in-complexity arithmetic operation is multiplication.
- The simplest multiplication is, of course, multiplication by 0 or 1, but such multiplication does not lead to any new natural numbers.
- The simplest multiplication that leads to new natural numbers is multiplication by 2.
- This leads to a binary number system, in which every natural number is represented:
  - as the sum of appropriate powers of 2,
  - i.e., as the sum of numbers obtained from 1 by consequent multiplication by 2.
- For example, $144 = 128 + 16 = 2^7 + 2^4$.
- Strictly speaking, this should be

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 \cdot 2.$$

20. What if we allow the simplest possible multiplication – multiplication by 2: binary numbers (cont-d)

- So, in effect, binary system uses also raising to the power.
- Specifically, it uses the simplest case of raising to the power, when we only raise number 2 to different powers.

21. What if we allow multiplication by several numbers: decimal numbers

- What if:
  - instead of multiplying by 2, we allow multiplying by all natural numbers from 2 to 10, and
  - instead of raising 2 to different powers we raise 10 to different powers.
- In this case, we get our usual decimal system, where, e.g., 144 is represented as

$$10 \cdot 10 + 4 \cdot 10 + 4 = 10^2 + 4 \cdot 10^1 + 4 \cdot 10^0.$$

- This is the first example where a natural sequence differs from a historic one:
  - decimal numbers were known for many centuries, since the 4th century or even earlier, while
  - binary numbers were only introduced by Leibniz in the 17 century.

22. How to represent fractions

- Let us now switch to describing how we can represent fractions.
- To represent fractions, we need to add division to operations describing natural numbers.
- As we have mentioned, in computers, division a/b is implemented as

$$a \cdot (1/b).$$

- So, a natural idea is to use the operation $1/b$.

23. What if we only allow division by 2: binary fractions

- As we have mentioned, for a computer, the simplest possible division is:
  - division by $b = 2$, and also
  - repeated divisions by 2, which are equivalent to dividing by the corresponding power of 2.
- Since we allow division by 2, it is reasonable to also allow similar simpler operations: multiplication by 2 and addition.
- This leads to binary fractions like

$$101.011_2 = 2^2 + 1 + 1/2^2 + 1/2^3.$$

24. What if we only allow division by 10: decimal fractions

- Instead of division by 2, we can allow division by 10.
- This operation is easy for us humans (as opposed to being easier for computers).
- In combination with multiplication by 10 (and multiplication by natural numbers < 10), this leads to our usual decimal fractions, e.g.:

$$21.44 = 2 \cdot 10 + 1 + 4/10 + 4/10^2.$$

- It is important to note that:
  - if we are only interested in proper fractions, i.e., fractions smaller than 1,
  - then we do not need multiplication;
  - all we need is to start with 9 digits $1, \ldots, 9$, and allow division by 10 and addition.
- For example, 0.44 can be represented as $0.44 = 4/10 + 4/10^2$.

25. What if we allow $1/b$ for all natural numbers b: Egyptian fractions

- In the previous two sections, we considered the cases when we only allow division by one specific natural number:
  - the number $b = 2$ for binary fractions and
  - the number $b = 10$ for decimal fraction.
- A natural next idea is to allow divisions by any natural number, i.e., to allow fractions of the type $1/b$, where b is an arbitrary integer.
- It turns out that:
  - if we are only interested in representing proper fractions,
  - then any such fractions can be represented as the sum of numbers $1/b$ of such type.
- For example, $2/3 = 1/2 + 1/6$, $3/4 = 1/2 + 1/4$, etc.
- Such representations are known as Egyptian fractions, since this is how ancient Egyptians represented fractions.

26. What if we allow $1/b$ for all natural numbers b and multiplications by all natural numbers: regular fractions

- What if, in addition to the inverse, instead of adding, we allow multiplication by any natural number?
- In this case, we get all possible fractions of the type a/b, i.e., all regular fractions like $2/3$ or $3/4$.

27. What if we allow $1/b$ for any numbers b: Kepler's fractions

- What if we allow $1/b$ for all possible numbers b?
- Of course:
  - if we start with a finite list of numbers and only apply the operation $1/b$,
  - then all we get is the original numbers and their inverses.
- So, to be able to represent arbitrary non-negative rational numbers, we need to allow at least one more operation.
- As we have mentioned earlier, the simplest arithmetic operation is adding 1.
- It turns out that:
  - if we start with 0 and use these two operations: the inverse $1/b$ and adding one,
  - then we can represent any non-negative fraction.

28. Kepler's fractions (cont-d)

- This idea was first proposed by Johannes Kepler in 1619.
- Thus, such fractions are known as Kepler fractions.
- This representation is also known as Calkin-Wilf tree.
- Interestingly, in this case we have a unique representation of each positive fraction.
- Indeed, if we have a fraction a / b and
  - the numerator and the denominator have a common factor,
  - then we can divide both a and b by this factor and get a fraction in which a and b have no common divisors.
- In this case, we cannot have $a = b$, so we must have either $a < b$ or $a > b$.
- If $a < b$, then $a/b < 1$, so we cannot get this number by adding 1 to a positive number.

29. Kepler's fractions (cont-d)

- Thus, we can get a/b by applying inverse to b/a.
- If $a > b$, then we can represent a/b as $(a-b)/b+1$.
- So to represent the number a/b this way, it is sufficient to represent the fraction $(a-b)/b$.
- In each such transformation, the maximum of numerator and denominator decreases.
- So in finitely many steps, this procedure will stop, and we will get the desired representation.
- Let us illustrate this idea on the examples of fractions $3/4$ and $5/13$.
- Let us start with $3/4$.
- Since $3/4 < 1$, we represent it as $1/(4/3)$.
- Here, $4/3 = 1/3 + 1$, i.e., $1/(1+1+1)+1$.
- Thus, $3/4 = 1/(1/(1+1+1)+1)$.

30. Kepler's fractions (cont-d)

- For $5/13$, we similarly have $5/13 = 1/(13/5)$.
- Here, $13/5 = 8/5 + 1$, and $8/5 = 3/5 + 1$, so $13/5 = 3/5 + 1 + 1$.
- In this case, $3/5 = 1/(5/3)$, where $5/3 = 2/3 + 1$.
- In its turn, $2/3 = 1/(3/2)$, where $3/2 = 1/2 + 1$, i.e.,

$$3/2 = 1/(1 + 1) + 1.$$

- So, $2/3 = 1/(1/(1 + 1) + 1)$, $5/3 = 1/(1 + 1/(1 + 1)) + 1$, thus

  $3/5 = 1/(1/(1/(1+1)+1)+1), 13/5 = 1/(1/(1/(1+1)+1)+1)+1+1,$

$$\text{and } 5/13 = 1/(1/(1/(1/(1 + 1) + 1) + 1) + 1 + 1).$$

31. Kepler's fractions (cont-d)

- Similar to Peano arithmetic, we can therefore describe non-negative rational numbers as follows:
  - 0 is a non-negative rational number;
  - if a is a non-negative rational number, then $a+1$ is a non-negative rational number, and
  - if a is a non-negative rational number different from 0, then $1/a$ is a non-negative rational number.

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