From Aristotle to Newton, from Sets to Fuzzy Sets, and from Sigmoid to ReLU: What Do All These Transitions Have in Common?

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1. Aristotelian physics: a brief reminder

- In ancient times, most researchers believed that a natural state of an un-disturbed object is when this object is not moving at all.
- In order to make the object move, we need to apply some force.
- If we stop applying this force, the object stops moving.
- Now we know that this description is wrong.
- However, from the commonsense viewpoint, this description makes perfect sense.
- For example, if we push a cart, it will start rolling, but once we stop pushing, it will pretty fast come to a stop.
- Because of this description, people believed that there is a force constantly pushing the planets and the Sun across the sky.
- For example, many religious people believed that angels are constantly pushing the Sun and the planets with their wings.

2. How can we describe this worldview in mathematical terms

- In mathematical terms, this description means that in order to change the location of an object, we need to apply some force.
- In other words, any change in the object's coordinates x has to be explained by an application of some force.
- From the mathematical viewpoint:
  - a change from the object's coordinate $x(t)$ at some moment of time t and its coordinate $x(t+\Delta t)$ at the "next" moment of time
  - can be described by the derivative:

$$\dot{x}(t) \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}.$$

- In these terms:
  - if no force is applied, the object does not move, so the derivative $\dot{x}(t)$ is equal to 0;
  - thus, the only way the derivative becomes different from 0 is when some external force is applied.

3. How can we describe this worldview in mathematical terms (cont-d)

- In other words, the derivative $\dot{x}$ is determined by the external forces.
- So we have an equation $\dot{x}(t) = F$, where F is the corresponding force.
- In mathematics, such equations – involving the first derivative of the function – are called *differential equations of first order*.

4. Galileo's discovery and the resulting Newtonian physics

- The Aristotelian approach explains the behavior of physical objects on the *qualitative* level.
- However:
  - when researchers started performing experiments,
  - they realized that this approach does not lead to a good *quantitative* explanation.
- One of the first researchers who experimentally tested effects of different forces on motion was Galileo.
- Based on his experiments, Galileo came up with a completely different approach to motion.

5. Galileo's discovery and the resulting Newtonian physics (cont-d)

- Namely, Galileo has discovered what he called *the Law of Inertia*:
  - if an object starts moving with a constant speed in some direction,
  - then, in the absence of external forces, it will continue this movement indefinitely, and
  - if the object stops moving, this means that some forces were applied.
- For example, when we stop pushing a cart, it does stop – because the force of friction is applied.
- And if we place a cart of a very smooth surface, where friction is small – e.g., on a smooth ice – it will continue rolling for a very long time.
- Similarly, if we throw a ball into the air, it will stop – because its motion is affected by the force of gravity.

6. Galileo's discovery and the resulting Newtonian physics (cont-d)

- Galileo provided quantitative description of some of the phenomena.
- However, the full description of motion came from Newton.
- Newton used the Law of Inertia as one of the main three laws of motion.
- However, he supplemented it with the other two laws.
- He was able to do it, since he invented calculus, the mathematical technique appropriate for describing motion.

7. How can we describe Newton's worldview in mathematical terms

- In mathematical terms, the above description means that in order to change the velocity of an object, we need to apply some force.
- In other words, any change in the object's velocity v has to be explained by an application of some force.
- From the mathematical viewpoint:
  - a change from the object's velocity $v(t)$ at some moment of time t and its velocity $v(t+\Delta t)$ at the "next" moment of time
  - can be described by the derivative:

$$\dot{v}(t) \approx \frac{v(t+\Delta t) - v(t)}{\Delta t}.$$

8. How can we describe Newton's worldview in mathematical terms (cont-d)

- In these terms:
  - if no force is applied, the object does not change its velocity, so the derivative $\dot{v}(t)$ is equal to 0;
  - thus, the only way the derivative becomes different from 0 is when some external force is applied.
- In other words, the derivative $\dot{v}$ is determined by the external forces.
- So we have an equation $\dot{v}(t) = F$, where F is the corresponding force.
- The velocity v itself is the derivative of the coordinates $v = \dot{x}$.
- So, its derivative $\dot{v}$ is the second derivative of the coordinate $\dot{v} = \ddot{x}$.
- Thus, the dynamic equation takes the form $\ddot{x} = F$.
- In mathematics, such equations – involving the second derivative of the function – are called *differential equations of second order*.

9. Why second order: a kind-of explanation

- So far:
  - the only argument that we provided in favor of second-other equations was
  - that they lead to a more accurate description of the physical world.
- But *why* do they provide a more accurate description of the physical world?
- A possible – although probably not very convincing – answer to this question comes from the fact that:
  - physical equations are usually described
  - in terms of an optimization principle.
- For example, light passing through several media follows the path for which the transition time is the smallest.
- Equations of motion can be equivalently described as saying that some functional $\int L(x, \dot{x})\, dt$ attains its smallest possible value.

10. Why second order: a kind-of explanation (cont-d)

- This functional is called *action*.
- It is known that the minimum of a function is attained when its derivative is equal to 0.
- Similarly, it can be shown that the minimum of a functional is attained when its so-called *functional derivative* is equal to 0.
- This derivative takes the form

$$\frac{\delta L}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial x \partial \dot{x}} \cdot \dot{x} - \frac{\partial^2 L}{\partial^2 \dot{x}} \cdot \ddot{x}.$$

- Equating this to 0 is always a second-order equation.
- In principle, we can consider functionals in which L depends only on x.
- But in this case, we simply have a static equation, no dynamics.

11. Why second order: a kind-of explanation (cont-d)

- We can also consider the case when L also depends on the second derivative.
- In this case, we get a 4-th order differential equation.
- Optimization never leads to the first- (or any odd-) order differential equations.
- The simplest even-order differential equation is the second-order one.

12. Case of ecology

- Interestingly, a similar transition, from first- to second-order differential equations, leads to more accurate models in ecology.
- In ecology, the above optimization-based explanation makes more sense than in physics.
- In physics, the fact that equations come from optimization principles is largely an empirical fact with no clear physical meaning.
- In biology, the need to have an optimal fit with the environment is one of the main principles.
- Without such fit, the species would not survive.

13. Other cases when optimality naturally appears

- There are other cases when optimality naturally appears: namely, the cases when we design objects or when we design algorithms.
- In all such cases, we want to select the design which is the best for our purpose.
- Let us consider two such examples.

14. Two examples: a general description

- In the first example, we are interested in how our degree of confidence $x(t)$ that a value t satisfies some property changes with t.
- This property may be "t is small", it may be "t is larger than 0", etc.
- In the second example, we are interested in the activation function $x(t)$ of a neuron, i.e., in a function that transforms:
  - a linear combination t of neuron's input signals
  - into the output signal.
- From the application viewpoint, these two examples deal with completely different situations.
- However, from the mathematical viewpoint, they are similar.
- In both cases, we are interested in a real-valued function $x(t)$ of a real-valued variable t.
- Let us analyze, for these two examples, what would descriptions in terms of first- and second-order differential equations lead to.

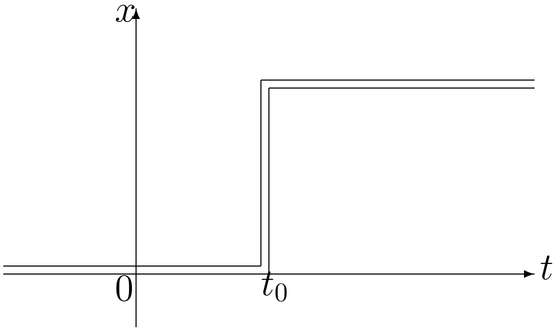
15. What will happen if we use first-order differential equations: mathematical analysis

- In general, as we have mentioned, a first-order differential equation has the form $\dot{x} = F$, for some function F describing force.
- In principle, at different moments of time, we can apply different forces.
- From this viewpoint, the basic, elementary effect is the application of a force which is active only at one specific moment of time.
- All other forces can be represented as combinations of such instantaneous forces applied at different moments of time.
- In mathematical terms, such an elementary force is described by a so-called delta-function $\delta(t - t_0)$.
- This function is equal to 0 for all other moments of time $t \neq t_0$.
- Let us therefore consider the first-order differential equation describing this force, i.e., the equation $\dot{x} = \delta(t - t_0)$.

16. What will happen if we use first-order differential equations: mathematical analysis (cont-d)

- For $t \neq t_0$, the derivative $\dot{x}$ of the function $x(t)$ is equal to 0.
- So both for $t < t_0$ and for $t > t_0$, the function $x(t)$ does not change.
- In other words, it is equal to a constant for $t < t_0$, and it is equal to another constant when $t > t_0$.
- It is reasonable to start with $x(t) = 0$, so for $t < t_0$, we have $x(t) = 0$.
- At the point t_0, the value of $x(t)$ jumps to some other value.
- So, we have the following step function $x(t)$:
  - for $t < t_0$, we have $x(t) = 0$, and
  - for $t > t_0$, we have $x(t) = c$ for some constant c.

17. What will happen if we use first-order differential equations: mathematical analysis (cont-d)



- Let us analyze what this means for both our examples.

18. Case of degrees of confidence

- In this case, we go from no confidence to the next level – which is natural to be associated with full confidence.
- This corresponds to the usual Boolean 2-valued logic, and this corresponds to sets – or, to be more precise, to infinite or final intervals.
- Specifically, when we have only one delta-function, we get an infinite interval.
- However, if we add another (negative) delta-function at the endpoint of the desired interval, we will get a finite interval as well.

19. Case of an activation function

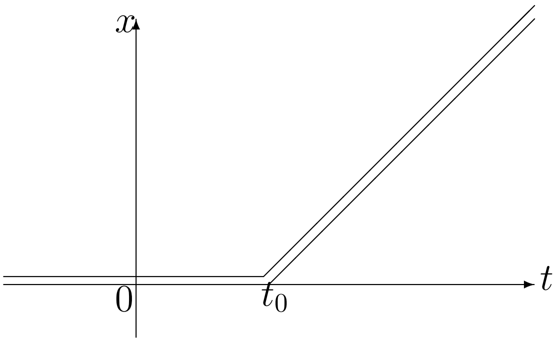
- For activation functions, a step-function is a good approximation to the *sigmoid* function.
- Sigmoid is the activation function that was, until recently, most widely used in artificial neural networks.

20. What will happen if we use second-order differential equations: mathematical analysis

- Second-order differential equations have the form $\ddot{x} = F$.
- As in the first-order case, the elementary force is a delta-function, so we get the equation $\ddot{x} = \delta(t - t_0)$.
- As we have mentioned earlier, this equation can be equivalently re-written as $\dot{v} = \delta(t - t_0)$, where we denoted $v \stackrel{\text{def}}{=} \dot{x}$.
- In terms of v, this is the same equation that we considered when we analyzed first-order equations.
- So, we can use the solution that we derived during that analysis.
- Namely, we conclude that:
  - for $t < t_0$, we have $\dot{x}(t) = v(t) = 0$, and
  - for $t > t_0$, we have $\dot{x}(t) = v(t) = c$ for some constant c.
- We know the derivative $\dot{x}(t)$ of the desired function $x(t)$.

21. What will happen if we use second-order differential equations: mathematical analysis (cont-d)

- So, to reconstruct this function, we can simply integrate the above expression.
- If we start with 0 as before, we get the following result:
  - for $t < t_0$, we have $x(t) = 0$, and
  - for $t > t_0$, we have $x(t) = c \cdot (t - t_0)$.



- Let us analyze what this means for both our examples.

22. What it means

- For degrees of confidence:
  - instead of only two possible values – as in the case of first-order equations,
  - we have degrees of confidence that continuously change from 0 to larger values.
- This is one of the main ideas behind *fuzzy logic*.
- Moreover, what we get is a linear shape of this increase.
- This is exactly what we get when we use *triangular membership functions* – the most widely used shape of membership functions.
- For activation functions, what we get is exactly *Rectified Linear Unit* (ReLU, for short).
- This is currently the most widely used activation function.

23. Conclusion

We showed that a natural optimization-motivated transition from first-order to second-order differential equations explains transitions:

- in physics,
- in describing degrees of confidence, and
- in describing activation functions in neural networks:

application area	first-order differential equations	second-order differential equations
physics	Aristotelian	Newtonian
degrees of confidence	sets, intervals	fuzzy sets, triangular membership functions
neural networks	sigmoid activation function	ReLU activation function

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