Log-Periodic Power Law as a Predictor of Catastrophic Events: A New Mathematical Justification

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1. Outline

- To decrease the damage caused by meteorological disasters, it is important to predict these disasters.
- In the vicinity of a catastrophic event, many parameters exhibit log-periodic power behavior.
- By fitting the formula to the observations, it is possible to predict the event.
- Log-periodic power behavior is observed in ruptures of fuel tanks, earthquakes, stock market disruptions, etc.
- In this talk, we provide a general system-based explanation of this law.
- This makes us confident that this law can be also used to predict meteorological disasters.

2. Formulation of the Problem

- To decrease the damage caused by meteorological disasters, it is important to predict these disasters.
- A natural idea is to see how similar disaster prediction problems are solved in other application areas.
- We need to predict mechanical disasters, earthquakes, financial disasters, etc.
- Some predictions comes from the observation that:
  - in the vicinity of a catastrophic event,
  - many parameters exhibit so-called log-periodic power behavior,
  - with oscillations of increasing frequency.
- Let us therefore describe this behavior in detail.

3. The Emergence of Log-Periodic Power Law in Disaster Prediction

- The history of log-periodic power law applications started with space exploration.
- To be able to safely return home, a spaceship needs to store fuel.
- A satellite is moving at a speed of 8 km/sec, much faster than the speediest bullet.
- At such a speed, a micro-meteorite or a piece of space debris can easily cause a catastrophic leak.
- To avoid such a bullet-type penetration, engineers use Kevlar, bulletproof material. Tests showed that:
  - while in general, Kevlar-coated tanks performed really well,
  - on a few occasions, the Kevlar tanks catastrophically exploded.

4. The Emergence of Log-Periodic Power Law in Disaster Prediction (cont-d)

- D. Sornette noticed that:
  - an explosion is usually preceded by oscillations;
  - their frequency increases as we approach the critical moment of time t_c.
- He observed that the dependence of each corresponding parameter x on time t has the form

$$x(t) = A + B \cdot (t_c - t)^z + C \cdot (t_c - t)^z \cdot \cos(\omega \cdot \ln(t_c - t) + \varphi).$$

- By fitting this model to the observations, we can predict the moment t_c of the catastrophic event.
- Sornette called the dependence (1) *Log-Periodic Power Law* (LPPL, for short).

5. Applications to Earthquake Prediction

- D. Sornette's wife, A. Sauron-Sornette, is also a scientist: she is a geophysicist.
- Naturally, the two scientist spouses talk about their research.
- From the mechanical viewpoint, an earthquake is simply a mechanical rupture.
- So, they decided to check whether the log-periodic power law occurs in earthquakes.
- In many cases, they observed the log-periodic power law behavior in the period preceding an earthquake.
- This technique is *not* a panacea: not all earthquakes can be this predicted.
- However, some can be predicted, and the ability to predict an earthquake decreases the damage.

6. Financial Applications

- With colleagues, D. Sornette observed similar log-periodic fluctuations before financial crashes.
- A similar observation was independently made by Feigenbaum and Freund.
- Both papers appeared in 1996 in physics journals, and were not widely understood by economists.
- In Summer 1997, D. Sornette and O. Ledoit used their techniques to predict the October 1997 market crash.
- By investing in put options, made a well-documented (and well-publicized) 400% profit on their investment.
- This caused attention of economists.
- Now log-periodic power law predictions are important part of the econometric toolbox.
- Not all financial crashes are predictable, but some are.

7. Towards a General Explanation for Log-Periodic Power Law

- Log-periodic power law is observed in different systems.
- This seems to indicate that the this law is caused by general properties of system.
- Some theoretical explanations have been published.
- However, these explanations are based on a very specific model of a system.
- It is desirable to come up with a more general explanation.
- This would make us confident that this law can also be used to predict meteorological disasters.

8. Analysis of the Problem

- We are interested in the dependence of quantities describing the system on time t: $x = x(t)$.
- In principle, we can have arbitrary functions $x(t)$.
- However, our objective is to make predictions by using appropriate computer models.
- In the computer, at any given moment of time, we can only represent finitely many parameters.
- It is therefore reasonable to consider finite-parametric families of functions $x(t) = f(c_1, \ldots, c_n, t)$.
- Usually, we know the approximate values $c_1^{(0)}, \ldots, c_n^{(0)}$ of the parameters c_i.

9. Analysis of the Problem (cont-d)

- In this case, the differences $\Delta c_i \stackrel{\text{def}}{=} c_i - c_i^{(0)}$ are small, so we can keep only linear terms in the Taylor expansion

$$x(t) = f(c_1, \ldots, c_n, t) = f(c_1^{(0)} + \Delta c_1, \ldots, c_n^{(0)} + \Delta c_n, t).$$

- So, $x(t) = f_0(t) + \Delta c_1 \cdot e_1(t) + \ldots + \Delta c_n \cdot e_n(t)$, where $f_0(t) \stackrel{\text{def}}{=} f(c_1^{(0)}, \ldots, c_n^{(0)}, t)$ and $e_i(t) \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial c_i}$.

- Substituting $\Delta c_i = c_i - c_i^{(0)}$ into this formula, we get

$$x(t) = e_0(t) + c_1 \cdot e_1(t) + \ldots + c_n \cdot e_n(t),$$

$$\text{where } e_0(t) \stackrel{\text{def}}{=} f_0(t) - \sum c_i^{(0)} \cdot e_i(t).$$

- In other words, the desired dependencies $x(t)$ are linear combinations of the appropriate functions $e_i(t)$.

10. Let Us Use Natural Symmetries

- To complete the description of the time dependence, we need to select the appropriate functions $e_i(t)$.
- To select these functions, we will use symmetry.
- The numerical value of time t depends:
  - on the choice of a starting point for measuring time;
  - on the choice of the measuring unit.
- If we replace the starting point with a one which is s_0 units earlier, then t changes to $t' = t + s_0$ (*shift*).
- Similarly, we can measure time in years or in days.
- If we replace the original unit of time with a one which is λ times smaller, then t changes to $t' = \lambda \cdot t$ (*scaling*).
- In general, t changes to $t' = \lambda \cdot t + s_0$.

11. How to Use Symmetries: Motivations and Definitions

- We want to find the general functions $e_i(t)$, functions which would be applicable to all kinds of phenomena.
- So, the resulting class of functions should not change if we change the starting point or the measuring unit.
- *By a* family of functions $\mathcal{F}$, *we mean a family consisting of all the functions of the type*

$$x(t) = e_0(t) + c_1 \cdot e_1(t) + \ldots + c_n \cdot e_n(t).$$

- *Example:* when $e_0(t) = 0$ and $e_i(t) = t^{i-1}$, $\mathcal{F}$ consists of all the polynomials of degree $\le n - 1$.
- *We say that a family of functions $\mathcal{F}$ is* shift- and scale-invariant *if for every $x(t) \in \mathcal{F}$ and for every λ and s_0,*

$$y(t) \in \mathcal{F}, \text{ where } y(t) \stackrel{\text{def}}{=} x(\lambda \cdot t + s_0).$$

12. First Result and Discussion

- **Proposition.** *Let $\mathcal{F}$ be a shift- and scale-invariant family; then all functions from $\mathcal{F}$ are polynomials.*
- This holds for processes with *no* special moment of time and *no* special time unit.
- For such processes, we can select different starting moments and different time units.
- If there is a catastrophic event, then its time t_c *is* a special moment.
- A natural starting moment of time is t_c, so a natural way to describe time is as a different $T \stackrel{\text{def}}{=} t_c - t$.
- We can still select different time units, so we still have scaling transformation $T \to T' = \lambda \cdot T$, i.e.,

$$t_c - t' = \lambda \cdot (t_c - t).$$

13. Main Result

- *We say that a family of functions $\mathcal{F}$ is* scale-invariant *if for every $x(T) \in \mathcal{F}$ and for every real λ, we have*

$$y(T) \in \mathcal{F}, \text{ where } y(T) \stackrel{\text{def}}{=} x(\lambda \cdot T).$$

- **Proposition.** *Let $\mathcal{F}$ be a scale-invariant family; then, all f-ns from $\mathcal{F}$ are linear combinations of functions*

$$T^z, \; T^z \cdot \cos(\omega \cdot \ln(T) + \varphi), \; T^z \cdot \cos(\omega \cdot \ln(T) + \varphi) \cdot (\ln(T))^k.$$

- Since $T = t_c - t$, we thus explain the semi-empirical formula

$$x(t) = A + B \cdot (t_c - t)^z + C \cdot (t_c - t)^z \cdot \cos(\omega \cdot \ln(t_c - t) + \varphi).$$

14. Which Functions Are Optimal?

- So far, we have proved that if the family $\mathcal{F}$ is invariant, then $\mathcal{F}$ consists of log-periodic power functions.
- Invariance definitely makes sense.
- However, a more natural idea is to select a family which is *optimal* – in some reasonable sense.
- When we say "optimal", we mean that there must be a relation $\succeq$ describing which family is better (or equal).
- This relation must be transitive:

$$\text{if } \mathcal{F} \succeq \mathcal{F}' \text{ and } \mathcal{F}' \succeq \mathcal{F}'', \text{ then } \mathcal{F} \succeq \mathcal{F}''.$$

- We would like to require that this relation be *final* in the sense that it should define a unique *optimal* $\mathcal{F}_{\rm opt}$:

$$\exists! \mathcal{F}_{\rm opt}\, \forall \mathcal{F}\, (\mathcal{F}_{\rm opt} \succeq \mathcal{F}).$$

15. Why Final Optimal Criterion?

- If none of the families is optimal, then this criterion is useless.
- If *several* different families are optimal, then we can use this ambiguity to optimize something else; e.g.:
  - if we have two families with the same approximating quality,
  - then we choose the one which is easier to compute.
- As a result, we get a new criterion: $\mathcal{F} \succeq_{\text{new}} \mathcal{F}'$ if
  - either $\mathcal{F}$ gives a better approximation,
  - or $\mathcal{F} \sim_{\text{old}} \mathcal{F}'$ and $\mathcal{F}$ is easier to compute.
- For this new criterion, the class of optimal families is narrower.
- We can repeat this procedure until we get a final criterion, for which there is only one optimal family.

16. An Optimality Criterion Should Be Invariant

- What is better in one representation should be better in another representation as well.
- In other words, it is reasonable to require the relation $\mathcal{F} \succeq \mathcal{F}'$ should be invariant w.r.t. $T \to T' = \lambda \cdot T$.
- *For every family of functions $\mathcal{F}$ and for every λ, by its* λ-rescaling $S_\lambda(\mathcal{F})$, *we mean* $\{x(\lambda \cdot T) : x(T) \in \mathcal{F}\}$:
  - if $\mathcal{F}$ consists of all the functions of the type
  $$e_0(T) + c_1 \cdot e_1(T) + \ldots + c_n \cdot e_n(T),$$
  - then $S_\lambda(\mathcal{F})$ consists of all the functions of the type
  $$e'_0(T) + c_1 \cdot e'_1(T) + \ldots + c_n \cdot e'_n(T),$$
  where $e'_i(T) \stackrel{\text{def}}{=} e_i(\lambda \cdot T)$.

17. Optimality: Definitions and Result

- Let $\mathcal{C}$ be a class of all families of functions.
- By an *optimality criterion*, we mean a *pre-ordering* (i.e., a transitive reflexive relation) $\preceq$ on the class $\mathcal{C}$.
- We say that $\preceq$ is *scale-invariant* if for all λ, and for all $\mathcal{F}, \mathcal{F}' \in \mathcal{C}$, $\mathcal{F} \preceq \mathcal{F}'$ implies $S_\lambda(\mathcal{F}) \preceq S_\lambda(\mathcal{F}')$.
- We say that $\preceq$ is *final* if there exists exactly one $\mathcal{F}_{\rm opt} \in \mathcal{C}$ which is preferable to all the others:

$$\exists! \mathcal{F}_{\rm opt}\, \forall \mathcal{F}\, (\mathcal{F} \preceq \mathcal{F}_{\rm opt}).$$

- **Proposition.** *Let $\preceq$ be scale-invariant and final; then, every $x(T) \in \mathcal{F}_{\rm opt}$ is a linear combination of the f-ns*

$$T^z,\ \ T^z\cdot\cos(\omega\cdot\ln(T)+\varphi),\ \ \textit{and}\ T^z\cdot\cos(\omega\cdot\ln(T)+\varphi)\cdot(\ln(T))^k.$$

18. Proof of the Scale-Invariance Result

- Scale-invariance means that if $x(T) \in \mathcal{F}$, then, for every λ, we have $x(\lambda \cdot T) \in \mathcal{F}$.
- Each $x(T) \in \mathcal{F}$ is a linear combinations of $a_0(T) = e_0(T)$ and $a_i(T) = e_0(T) + e_i(T)$, $i \geq 1$.
- So, it is sufficient to require this for $a_i(T)$:

$$a_i(\lambda \cdot T) = k_{i0}(\lambda) \cdot a_0(T) + \ldots + k_{in}(\lambda) \cdot a_n(T) \text{ for some } k_{ij}(\lambda).$$

- For each i, we select $n+1$ different values T_k, and get $n+1$ linear equations for $n+1$ unknowns $k_{ij}(\lambda)$:

$$a_i(\lambda \cdot T_k) = k_{i0}(\lambda) \cdot a_0(T_k) + \ldots + k_{in}(\lambda) \cdot a_n(T_k).$$

- Cramer's rule describes $k_{ij}(\lambda)$ as a differentiable function of $a_i(\lambda \cdot T_k)$ and $a_i(T_k)$.
- Since the functions $e_i(T)$ are differentiable, we conclude that $a_i(T)$ and $k_{ij}(\lambda)$ are differentiable as well.

19. Proof of the Scale-Invariance Result (cont-d)

- Differentiating by λ and taking $\lambda = 1$, we get
$$T \cdot \frac{da_i}{dT} = C_{i0} \cdot a_0(T) + \ldots + C_{in} \cdot a_n(T).$$
- For $S \stackrel{\text{def}}{=} \ln(T)$, we have $\frac{dT}{T} = dS$, so
$$\frac{da_i}{dS} = C_{i0} \cdot a_0(S) + \ldots + C_{in} \cdot a_n(S).$$
- This is a system of linear differential equations with constant coefficients.
- A general solution to such a system is a linear combination of
$$\exp(z \cdot S), \quad S^k \cdot \exp(z \cdot S), \quad \exp(z \cdot S) \cdot \cos(\omega \cdot S + \varphi),$$
$$\text{and } S^k \cdot \exp(z \cdot S) \cdot \cos(\omega \cdot S + \varphi).$$
- Substituting $S = \ln(T)$, we get the desired expressions.

20. Proof of the Shift-Invariance Result

- Shift-invariance means that if $x(t) \in \mathcal{F}$, then, for every s_0, we have $x(t+s_0) \in \mathcal{F}$.
- Since each function $x(t) \in \mathcal{F}$ is a linear combinations of $a_i(t)$, it is sufficient to require this for $a_i(t)$:

$$a_i(t+s_0) = s_{i0}(s_0) \cdot a_0(t) + \ldots + s_{in}(s_0) \cdot a_n(t) \text{ for some } s_{ij}(s_0).$$

- For each i, we select $n+1$ different values t_k, and get $n+1$ linear equations for $n+1$ unknowns $s_{ij}(s_0)$:

$$a_i(t_k+s_0) = s_{i0}(s_0) \cdot a_0(t_k) + \ldots + s_{in}(s_0) \cdot a_n(t_k).$$

- Cramer's rule describes $s_{ij}(s_0)$ as a differentiable function of $a_i(t_k+s_0)$ and $a_i(t_k)$.
- Since the functions $e_i(t)$ are differentiable, we conclude that $a_i(t)$ and $s_{ij}(s_0)$ are differentiable as well.

21. Proof of the Shift-Invariance Result (cont-d)

- $a_i(t_k + s_0) = s_{i0}(s_0) \cdot a_0(t_k) + \ldots + s_{in}(s_0) \cdot a_n(t_k)$.
- Differentiating by s_0 and taking $s_0 = 0$, we get linear differential equations with constant coefficients:
$$\frac{da_i}{dt} = S_{i0} \cdot a_0(t) + \ldots + S_{in} \cdot a_n(t).$$
- So $a_i(t)$ is a linear combination of $\exp(z \cdot t)$, $t^k \cdot \exp(z \cdot t)$, $\exp(z \cdot t) \cdot \cos(\omega \cdot t + \varphi)$, and $t^k \cdot \exp(z \cdot t) \cdot \cos(\omega \cdot t + \varphi)$.
- Since $\mathcal{F}$ is also scale-invariant, each $a_i(t)$ is also a linear combination of t^z, $t^z \cdot (\ln(t))^k$, $t^z \cdot \cos(\omega \cdot \ln(t) + \varphi)$, and
$$t^z \cdot \cos(\omega \cdot \ln(t) + \varphi) \cdot (\ln(t))^k.$$
- The only functions which can be described as linear combinations w.r.t. both lists are polynomials $\sum c_k \cdot t^k$.
- Each $x(t) \in \mathcal{F}$ is a linear combination of polynomials, thus a polynomial.

22. Proof of the Optimization Result

- Since the criterion $\preceq$ is final, there exists exactly one optimal family; let us denote it by $\mathcal{F}_{\rm opt}$.
- Let us show that $S_\lambda(\mathcal{F}_{\rm opt}) = \mathcal{F}_{\rm opt}$ for every λ.
- Indeed, from the optimality of $\mathcal{F}_{\rm opt}$, we conclude that for every $\mathcal{F} \in \mathcal{C}$, we have $S_{1/\lambda}(\mathcal{F}) \preceq \mathcal{F}_{\rm opt}$.
- From the scale-invariance of $\preceq$ and from the fact that $S_\lambda(S_{1/\lambda}(\mathcal{F})) = \mathcal{F}$, we conclude that $\mathcal{F} \preceq S_\lambda(\mathcal{F}_{\rm opt})$.
- This is true for all $\mathcal{F} \in \mathcal{C}$ and therefore, the family $S_\lambda(\mathcal{F}_{\rm opt})$ is optimal.
- But since the optimality criterion is final, there is only one optimal family; hence, $S_\lambda(\mathcal{F}_{\rm opt}) = \mathcal{F}_{\rm opt}$.
- The result now follows from the scale-invariance proposition.

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