Kinematic Spaces: Motivations from Space-Time Physics, Main Mathematical Results, Algorithmic Results and Challenges, and Possible Relation to de Vries Algebras

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1. Why Ordering Relations

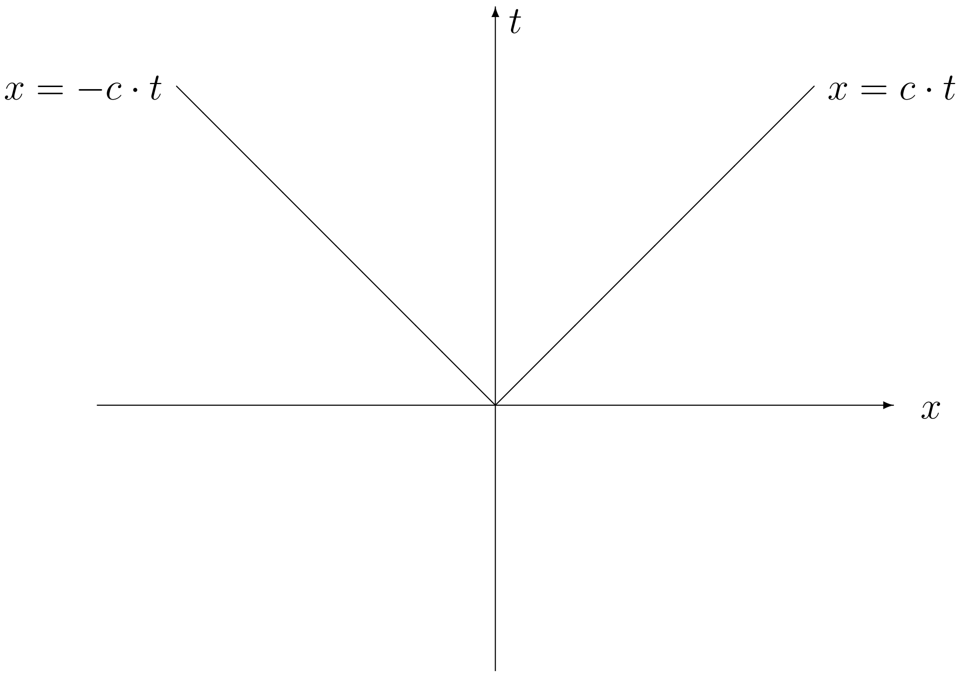
- Traditionally, in physics, space-times are described by (pseudo-)Riemann spaces, i.e.:
  - by smooth manifolds
  - with a tensor metric field $g_{ij}(x)$.
- However, in several physically interesting situations smoothness is violated and metric is undefined:
  - near the singularity (Big Bang),
  - at the black holes, and
  - on the microlevel, when we take into account quantum effects.
- In all these situations, what remains is causality $\preccurlyeq$ – an ordering relation.
- Geometers H. Busemann, R. Pimenov, physicists E. Kronheimer, R. Penrose: a theory of kinematic spaces.

2. Causality: Brief History

- In Newton's physics, signals can potentially travel with an arbitrarily large speed.
- Let $a = (t, x)$ denote an event occurring at the spatial location x at time t.
- Then, an event $a = (t, x)$ can influence an event $a' = (t', x')$ if and only if $t \le t'$.
- The fundamental role of the non-trivial causality relation emerged with the Special Relativity (SRT).
- In SRT, the speed of all the signals is limited by the speed of light c.
- As a result, $a = (t, x) \preccurlyeq a' = (t', x')$ if and only if $t' \ge t$ and $\dfrac{d(x, x')}{t' - t} \le c$, i.e.:

$$c \cdot (t' - t) \ge \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}.$$

3. Causality: A Graphical Description



4. Importance of Causality

- In the original special relativity theory, causality was just one of the concepts.
- Its central role was revealed by A. D. Alexandrov (1950) who showed that in SRT, causality implied Lorenz group:
- Every order-preserving transforming of the corr. partial ordered set is linear, and is a composition of:
  - spatial rotations,
  - Lorentz transformations (describing a transition to a moving reference frame), and
  - re-scalings $x \to \lambda \cdot x$ (corresponding to a change of unit for measuring space and time).
- This theorem was later generalized by E. Zeeman and is known as the *Alexandrov-Zeeman theorem*.

5. When is Causality Experimentally Confirmable?

- In many applications, we only observe an event b with some accuracy.
- For example, in physics, we may want to check what is happening exactly 1 second after a certain reaction.
- However, in practice, we cannot measure time exactly, so, we observe an event occurring 1 ± 0.001 sec after a.
- In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b.
- Because of this uncertainty, the only possibility to experimentally confirm that a can influence b is when

$$a \prec b \Leftrightarrow \exists U_b \, \forall \widetilde{b} \in U_b \, (a \preccurlyeq \widetilde{b}).$$

- In topological terms, this means that b is in the interior K_a^+ of the closed cone $C_a^+ = \{c : a \preccurlyeq c\}$.

6. Kinematic Orders

- In physics, $a \prec b$ correspond to influences with speeds smaller than the speed of light.
- There are two types of objects:
  - objects with non-zero rest mass can travel with any possible speed $v < c$ but not with the speed c;
  - objects with zero rest mass (e.g., photons) can travel only with the speed c, but not with $v < c$.
- Thus, $\prec$ correspond to causality by traditional (kinematic) objects.
- Because of this:
  - the relation $\prec$ is called *kinematic causality*, and
  - spaces with this relation $\prec$ are called *kinematic spaces*.

7. Kinematic Spaces: Towards a Description

- To describe space-time, we thus need a (pre-)ordering relation $\preccurlyeq$ (causality) and topology (= closeness).
- Natural continuity: for every event a and for every neighborhood U_a, there exist $a^- \prec a$ and $a \prec a^+$.
- Natural topology: every neighborhood U_a contains an open interval $(a', a'') = \{b : a' \prec c \prec a''\}$.
- Natural idea: a motion with speed c is a limit of motions with speeds $v < c$ when $v \to c$.
- Resulting description of $\preccurlyeq$ in terms of $\prec$: $C^+ = \overline{K^+}$ and $C^- = \overline{K^-}$, i.e., $b \succcurlyeq a \Leftrightarrow \forall U_b\, \exists \widetilde{b}\, \left(\widetilde{b} \in U_b \,\&\, \widetilde{b} \succ a\right)$.
- For $U_b = (b', b'')$, when $b \prec b''$, we get $a \prec \widetilde{b} \prec b''$ hence $a \prec b''$.
- Thus, $a \preccurlyeq b \Leftrightarrow \forall c (b \prec c \Rightarrow a \prec c)$.

8. Resulting Definition

- A set X with a partial order $\prec$ is called a *kinematic space* if is satisfies the following conditions:

$$\forall a\, \exists a_-, a_+\, (a_- \prec a \prec a_+);$$

$$\forall a, b\, (a \prec b \to \exists c\, (a \prec c \prec b));$$

$$\forall a, b, c\, (a \prec b, c \to \exists d\, (a \prec d \prec b, c));$$

$$\forall a, b, c\, (b, c \prec a \to \exists d\, (b, c \prec d \prec a)).$$

- We take a topology generated by intervals

$$(a, b) = \{c : a \prec c \prec b\}.$$

- A kinematic space is called *normal* if

$$b \in \overline{\{c : c \succ a\}} \Leftrightarrow a \in \overline{\{c : c \prec b\}}.$$

- For a normal kinematic space, we denote $b \in \overline{\{c : c \succ a\}}$ by $a \preccurlyeq b$.
- It is proven that $a \prec b \preccurlyeq c$ and $a \preccurlyeq b \prec c$ imply $a \prec c$.

9. First Result: Reconstructing $\prec$ from $\preccurlyeq$

- We consider separable kinematic spaces.
- We say that a space is *complete* if every $\preccurlyeq$-decreasing bounded sequence $\{s_n\}$ has a limit, i.e., $\wedge s_n$.
- *Lemma.* If every closed intervals $\{c : a \preccurlyeq c \preccurlyeq b\}$ is compact, then the space is complete.
- If two complete separable normal kinematic orders $\prec$ and $\prec'$ on X lead to the same closed order $\preccurlyeq = \preccurlyeq'$, then

$$\prec = \prec' .$$

- Let S_e denote the set of all $\preccurlyeq$-decreasing sequences $s = \{s_n\}$ for which $\wedge s_n = e$.
- For $s, s' \in S_e$, we define $s \geq s' \Leftrightarrow \forall n \, \exists m (s_n \succcurlyeq s'_m)$; then:

$$a \succ b \Leftrightarrow \exists e \succcurlyeq b \, \exists s = \{s_n\} \in S_e \, (s \text{ is largest in } S_e \,\&\, s_1 = a).$$

10. Symmetry: a Fundamental Property of the Physical World

- One of the main objectives of science: prediction.
- *Basis for prediction:* we observed similar situations in the past, and we expect similar outcomes.
- *In mathematical terms:* similarity corresponds to symmetry, and similarity of outcomes – to invariance.
- *Example:* we dropped the ball, it fall down.
- *Symmetries:* shift, rotation, etc.
- *In modern physics:* theories are usually formulated in terms of symmetries (not diff. equations).
- *Natural idea:* let us use symmetry to analyze causality as well.

11. Static Space-Times: Space-Times Invariant under Arbitrary Temporal Shifts

- Let $\{T_t\}$, $t \in \mathbb{R}$, be a 1-parametric group of $\preccurlyeq$-preserving transformations on a (pre-)ordered set $(E, \preccurlyeq)$.
- We require that:
  - if $t > 0$, then $\forall e\,(e \preccurlyeq T_t(e)\,\&\,T_t(e) \not\preccurlyeq e)$;
  - if $t_n \to t$ and $\forall n\,(e \preccurlyeq T_{t_n}(e'))$, then $e \preccurlyeq T_t(e')$;
  - for every $e, e' \in E$, there exists a t for which $e \preccurlyeq T_t(e')$ (no *cosmological (particle) horizons*).
- Then, there exists a set X with a function $d : X \times X \to \mathbb{R}$ for which $E \approx \mathbb{R} \times X$ with
$$(t, x) \preccurlyeq (t', x') \Leftrightarrow t' - t \geq d(x, x').$$
- We can have a metric space (X, d) $\Leftrightarrow$ the space E is T-*invariant* w.r.t. some $T : E \to E$ s.t. $T^2 = \text{id}$ and
$$a \preccurlyeq b \Leftrightarrow T(b) \preccurlyeq T(a).$$

12. de Vries Algebras: Definition

A *de Vries algebra* is a pair consisting of a complete Boolean algebra $(B, \preccurlyeq)$ and a binary relation $\prec$ (*proximity*) for which:

- $1 \prec 1$;
- $a \prec b$ implies $a \preccurlyeq b$;
- $a \preccurlyeq b \prec c \preccurlyeq d$ implies $a \prec d$;
- $a \prec b, c$ implies $a \prec b \wedge c$;
- $a \prec b$ implies $\neg b \prec \neg a$;
- $a \prec b$ implies there exists c such that $a \prec c \prec b$;
- $a \neq 0$ implies there exists $b \neq 0$ such that $b \prec a$.

13. Relation Between Kinematics and de Vries Algebras

- We say that a de Vries algebra is *connected* if $a \prec a$ implies that $a = 0$ or $a = 1$.
- For every connected de Vries algebra B:
  - the set $B - \{0, 1\}$ with a proximity relation $\prec$ is a normal kinematic space, and
  - the original relation $\preccurlyeq$ coincides with the closure of $\prec$ in the sense of kinematic spaces.
- Let S be a normal kinematic space with anti-tonic mapping $\neg$ for which $\neg\neg a = a$ and for which,
  - if we add 0 and 1 to the corresponding set $(S, \preccurlyeq)$,
  - we get a complete Boolean algebra.

  Then, this set is also a connected de Vries algebra.

14. Product Operations for Posets: Examples

- Let us assume that we have two parallel (independent) universes A_1 and A_2.
- Then an event in a multi-verse is a pair (a_1, a_2), where $a_1 \in A_1$ and $a_2 \in A_2$.
- To compare such pairs, we must therefore define a partial order on the set $A_1 \times A_2$ of all such pairs.
- For independent universes, a natural definition is a *Cartesian* product:

$$(a_1, a_2) \le (a'_1, a'_2) \Leftrightarrow ((a_1 \le a'_1) \,\&\, (a_2 \le a'_2)).$$

- Another operation is a *lexicographic* product:

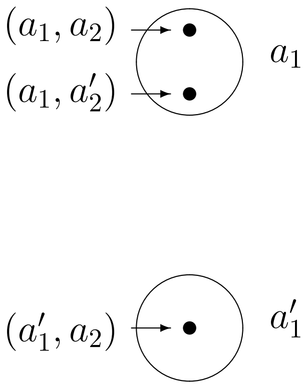
$$(a_1, a_2) \le (a'_1, a'_2) \Leftrightarrow$$

$$((a_1 \le a'_1) \,\&\, a_1 \ne a'_1) \vee ((a_1 = a'_1) \,\&\, (a_2 \le a'_2))).$$

15. Possible Physical Meaning of Lexicographic Order

Idea:

- A_1 is *macroscopic* space-time,
- A_2 is *microscopic* space-time:



16. Natural Questions

- *Question:* when does the resulting partially ordered set $A_1 \times A_2$ satisfy a certain property?
- *Examples:* is it a total order? is it a lattice order?
- *It is desirable* to reduce the question about $A_1 \times A_2$ to questions about properties of component spaces A_i.
- *Some such reductions are known*; e.g.:
  - A Cartesian product is a total order $\Leftrightarrow$ one of A_i is a total order, and the other has only one element.
  - A lexicographic product is a total order if and only if both components are totally ordered.
- *In this talk*, we provide a general algorithm for such reduction.

17. Similar Questions in Other Areas

- Similar questions arise in *other applications* of ordered sets.
- *Our algorithm* does not use the fact that the original relations are orders.
- Thus, our algorithm is applicable to a *general* binary *relation* – equivalence, similarity, etc.
- Moreover, this algorithm can be applied to the case when we have a space with *several* binary relations.
- *Example:* we may have an order relation and a similarity relation.

18. Definitions

- *By a* space, *we mean a set A with m binary relations $P_1(a,a')$, ..., $P_m(a,a')$.*
- *By a* 1st order property, *we mean a formula F obtained from $P_i(x,x')$ by using logical $\vee$, &, $\neg$, $\rightarrow$, $\exists x$ and $\forall x$.*
- *Note:* most properties of interest are 1st order; e.g. to be a total order means $\forall a \forall a' ((a \le a') \vee (a' \le a))$.
- *By a* product operation, *we mean a collection of m propositional formulas that*
  - *describe the relation $P_i((a_1,a_2),(a'_1,a'_2))$ between the elements $(a_1,a_2),(a'_1,a'_2) \in A_1 \times A_2$*
  - *in terms of the relations between the components $a_1, a'_1 \in A_1$ and $a_2, a'_2 \in A_2$ of these elements.*
- *Note:* both Cartesian and lexicographic order are product operations in this sense.

19. Main Result

- **Main Result.** *There exists an algorithm that, given*
  - *a product operation and*
  - *a property F,*

  generates a list of properties F_{11}, F_{12}, ..., F_{p1}, F_{p2} s.t.:

$$F(A_1 \times A_2) \Leftrightarrow ((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

- *Example:* For Cartesian product and total order F, we have

  $$F(A_1 \times A_2) \Leftrightarrow ((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee (F_{21}(A_1) \,\&\, F_{22}(A_2))) :$$

  - $F_{11}(A_1)$ means that A_1 is a total order,
  - $F_{12}(A_2)$ means that A_2 is a one-element set,
  - $F_{21}(A_1)$ means that A_1 is a one-element set, and
  - $F_{22}(A_2)$ means that A_2 is a total order.

20. Auxiliary Results

- *Generalization:*
  - A similar algorithm can be formulated for a product of three or more spaces.
  - A similar algorithm can be formulated for the case when we allow ternary and higher order operations.
- *Specifically for partial orders:*
  - The only product operations that always leads to a partial order on $A_1 \times A_2$ for which

    $$(a_1 \le_1 a'_1 \,\&\, a_2 \le_2 a'_2) \to (a_1, a_2) \le (a'_1, a'_2)$$

    are Cartesian and lexicographic products.

21. Proof of the Main Result

- The desired property $F(A_1 \times A_2)$ uses:
  - relations $P_i(a, a')$ between elements $a, a' \in A_1 \times A_2$;
  - quantifiers $\forall a$ and $\exists a$ over elements $a \in A_1 \times A_2$.
- Every element $a \in A_1 \times A_2$ is, by definition, a pair (a_1, a_2) in which $a_1 \in A_1$ and $a_2 \in A_2$.
- Let us explicitly replace each variable with such a pair.
- By definition of a product operation:
  - each relation $P_i((a_1, a_2), (a'_1, a'_2))$
  - is a propositional combination of relations betw. elements $a_1, a'_1 \in A_1$ and betw. elements $a_2, a'_2 \in A_2$.
- Let us perform the corresponding replacement.
- Each quantifier can be replaced by quantifiers corresponding to components: e.g., $\forall (a_1, a_2) \Leftrightarrow \forall a_1 \forall a_2$.

22. Proof of the Main Result (cont-d)

- So, we get an equivalent reformulation of F s.t.:
  - elementary formulas are relations between elements of A_1 or between A_2, and
  - quantifiers are over A_1 or over A_2.
- We use induction to reduce to the desired form

$$((F_{11}(A_1)\,\&\,F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1)\,\&\,F_{p2}(A_2))).$$

- Elementary formulas are already of the desired form – provided, of course, that we allow free variables.
- We will show that:
  - if we apply a propositional connective or a quantifier to a formula of this type,
  - then we can reduce the result again to the formula of this type.

23. Applying Propositional Connectives

- We apply propositional connectives to formulas of the type

$$((F_{11}(A_1) \,\&\, F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1) \,\&\, F_{p2}(A_2))).$$

- We thus get a propositional combination of the formulas of the type $F_{ij}(A_j)$.
- An arbitrary propositional combination can be described as a disjunction of conjunctions (DNF form).
- Each conjunction combines properties related to A_1 and properties related to A_2, i.e., has the form

$$G_1(A_1) \,\&\, \ldots \,\&\, G_p(A_1) \,\&\, G_{p+1}(A_2) \,\&\, \ldots \,\&\, G_q(A_2).$$

- Thus, each conjunction has the from $G(A_1) \,\&\, G'(A_2)$, where $G(A_1) \Leftrightarrow (G_1(A_1) \,\&\, \ldots \,\&\, G_p(A_1))$.
- Thus, the disjunction of such properties has the desired form.

24. Applying Existential Quantifiers

- When we apply $\exists a_1$, we get a formula
$$\exists a_1\,((F_{11}(A_1)\,\&\,F_{12}(A_2))\vee\ldots\vee(F_{p1}(A_1)\,\&\,F_{p2}(A_2))).$$
- It is known that $\exists a\,(A\vee B)$ is equivalent to $\exists a\,A\vee\exists a\,B$.
- Thus, the above formula is equivalent to a disjunction
$$\exists a_1\,(F_{11}(A_1)\,\&\,F_{12}(A_2))\vee\ldots\vee\exists a_1\,(F_{p1}(A_1)\,\&\,F_{p2}(A_2)).$$
- Thus, it is sufficient to prove that each formula $\exists a_1\,(F_{i1}(A_1)\,\&\,F_{i2}(A_2))$ has the desired form.
- The term $F_{i2}(A_2)$ does not depend on a_1 at all, it is all about elements of A_2.
- Thus, the above formula is equivalent to
$$(\exists a_1\,F_{i1}(A_1))\,\&\,F_{i2}(A_2).$$
- So, it is equivalent to the formula $F'_{i1}(A_1)\,\&\,F_{i2}(A_2)$, where $F'_{i1}\Leftrightarrow\exists a_1\,F_{i1}(A_1)$.

25. Applying Universal Quantifiers

- When we apply a universal quantifier, e.g., $\forall a_1$, then we can use the fact that $\forall a_1\, F$ is equivalent to $\neg\exists a_1\, \neg F$.
- We assumed that the formula F is of the desired type

$$(F_{11}(A_1)\,\&\,F_{12}(A_2)) \vee \ldots \vee (F_{p1}(A_1)\,\&\,F_{p2}(A_2)).$$

- By using the propositional part of this proof, we conclude that $\neg F$ can be reduced to the desired type.
- Now, by applying the $\exists$ part of this proof, we conclude that $\exists a_1\, (\neg F)$ can also be reduced to the desired type.
- By using the propositional part again, we conclude that $\neg(\exists a_1\, \neg F)$ can be reduced to the desired type.
- By induction, we can now conclude that the original formula can be reduced to the desired type.
- The main result is proven.

26. Example of Applying the Algorithm

- Let us apply our algorithm to checking whether a Cartesian product is totally ordered.
- In this case, F has the form $\forall a \forall a' \,((a \le a') \vee (a' \le a))$.
- We first replace each variable $a, a' \in A_1 \times A_2$ with the corresponding pair:

$$\forall (a_1, a_2) \forall (a'_1, a'_2)\,(((a_1, a_2) \le (a'_1, a'_2)) \vee ((a'_1, a'_2) \le (a_1, a_2))).$$

- Replacing the ordering relation on the Cartesian product with its definition, we get

$$\forall (a_1, a_2) \forall (a'_1, a'_2)\,((a_1 \le a'_1 \,\&\, a_2 \le a'_2) \vee (a'_1 \le a_1 \,\&\, a'_2 \le a_2)).$$

- Replacing $\forall a$ over pairs with individual $\forall a_i$, we get:

$$\forall a_1 \forall a_2 \forall a'_1 \forall a'_2\,((a_1 \le a'_1 \,\&\, a_2 \le a'_2)) \vee ((a'_1 \le a_1 \,\&\, a'_2 \le a_2))).$$

- By using the $\forall \Leftrightarrow \neg \exists \neg$, we get an equivalent form

$$\neg \exists a_1 \exists a_2 \exists a'_1 \exists a'_2\, \neg((a_1 \le a'_1 \,\&\, a_2 \le a'_2) \vee (a'_1 \le a_1 \,\&\, a'_2 \le a_2))).$$

27. Example (cont-d)

- So far, we got:

$$\neg\exists a_1\exists a_2\exists a_1'\exists a_2'\,\neg((a_1 \le a_1' \,\&\, a_2 \le a_2')\vee(a_1' \le a_1 \,\&\, a_2' \le a_2))).$$

- Moving $\neg$ inside the propositional formula, we get

$$\neg\exists a_1\exists a_1\exists a_1'\exists a_2'\,((a_1 \not\le a_1' \vee a_2 \not\le a_2')\,\&\,(a_1' \not\le a_1 \vee a_2' \not\le a_2))).$$

- The formula $(a_1 \not\le a_1' \vee a_2 \not\le a_2'))\,\&\,(a_1' \not\le a_1 \vee a_2' \not\le a_2)$ must now be transformed into a DNF form.

- The result is $(a_1 \not\le a_1' \,\&\, a_1' \not\le a_1)\vee(a_1 \not\le a_1' \,\&\, a_2' \not\le a_2)\vee (a_2 \not\le a_2' \,\&\, a_1' \not\le a_1) \vee (a_2 \not\le a_2' \,\&\, a_2' \not\le a_2)$.

- Thus, our formula is $\Leftrightarrow \neg(F_1 \vee F_2 \vee F_3 \vee F_4)$, where

$$F_1 \Leftrightarrow \exists a_1\exists a_2\exists a_1'\exists a_2'\,(a_1 \not\le a_1' \,\&\, a_1' \not\le a_1),$$

$$F_2 \Leftrightarrow \exists a_1\exists a_2\exists a_1'\exists a_2'\,(a_1 \not\le a_1' \,\&\, a_2' \not\le a_2),$$

$$F_3 \Leftrightarrow \exists a_1\exists a_2\exists a_1'\exists a_2'\,(a_2 \not\le a_2' \,\&\, a_1' \not\le a_1),$$

$$F_4 \Leftrightarrow \exists a_1\exists a_2\exists a_1'\exists a_2'\,(a_2 \not\le a_2' \,\&\, a_2' \not\le a_2).$$

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 28 of 32
Go Back
Full Screen
Close
Quit

28. Example (cont-d)

- So far, we got $\Leftrightarrow \neg(F_1 \vee F_2 \vee F_3 \vee F_4)$, where

$$F_1 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_1 \not\leq a_1' \,\&\, a_1' \not\leq a_1),$$

$$F_2 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_1 \not\leq a_1' \,\&\, a_2' \not\leq a_2),$$

$$F_3 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_2 \not\leq a_2' \,\&\, a_1' \not\leq a_1),$$

$$F_4 \Leftrightarrow \exists a_1 \exists a_2 \exists a_1' \exists a_2' \, (a_2 \not\leq a_2' \,\&\, a_2' \not\leq a_2).$$

- By applying the quantifiers to the corresponding parts of the formulas, we get

$$F_1 \Leftrightarrow \exists a_1 \exists a_1' \, (a_1 \not\leq a_1' \,\&\, a_1' \not\leq a_1),$$

$$F_2 \Leftrightarrow (\exists a_1 \exists a_1' \, a_1 \not\leq a_1') \,\&\, (\exists a_2 \exists a_2' \, a_2' \not\leq a_2),$$

$$F_3 \Leftrightarrow (\exists a_1 \exists a_1' \, a_1' \not\leq a_1) \,\&\, (\exists a_2 \exists a_2' \, a_2 \not\leq a_2'),$$

$$F_4 \Leftrightarrow \exists a_2 \exists a_1' \exists a_2' \, (a_2 \not\leq a_2' \,\&\, a_2' \not\leq a_2).$$

- Then, we again reduce $\neg(F_1 \vee F_2 \vee F_3 \vee F_4)$ to DNF.

29. Products of Ordered Sets: What Is Known

- At present, two product operations are known:
  - *Cartesian* product
$$(a_1, a_2) \le (a'_1, a'_2) \Leftrightarrow (a_1 \le_1 a'_1 \,\&\, a_2 \le_2 a'_2);$$
    and
  - *lexicographic* product
$$(a_1, a_2) \le (a'_1, a'_2) \Leftrightarrow$$
$$((a_1 \le_1 a'_1 \,\&\, a_1 \ne a'_1) \vee (a_1 = a'_1 \,\&\, a_2 \le_2 a'_2).$$
- *Question:* what other operations are possible?

30. Theorem

- By a *product operation*, we mean a Boolean function

$$P : \{T, F\}^4 \to \{T, F\}.$$

- For every two partially ordered sets A_1 and A_2, we define the following relation on $A_1 \times A_2$:

$$(a_1, a_2) \le (a_1', a_2') \stackrel{\text{def}}{=}$$

$$P(a_1 \le_1 a_1', a_1' \le_1 a_1, a_2 \le_2 a_2', a_2' \le_2 a_2).$$

- We say that a product operation is *consistent* if $\le$ is always a partial order, and

$$(a_1 \le_1 a_1' \,\&\, a_2 \le_2 a_2') \Rightarrow (a_1, a_2) \le (a_1', a_2').$$

- **Theorem:** *Every consistent product operation is the Cartesian or the lexicographic product.*

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