Towards Algebraic Foundations of Algebraic Fuzzy Logic Operations: Aiming at the Minimal Number of Requirements

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1. Background. I. Why Fuzzy Logic

- In many applications, it is important to use *expert knowledge*.
- Experts often describe their knowledge in *imprecise* ("*fuzzy*") properties like "small".
- *Example* of imprecision: for a specific size, an expert may be not fully confident whether this size is small.
- To describe such properties, *fuzzy logic* was invented.
- In fuzzy logic, each statement is characterized by a *degree* of confidence.
- Usually, this degree is taken from the interval $[0, 1]$, where:
  - 0 means absolutely false and
  - 1 means absolutely true.

2. Background. II. Fuzzy Logic Operations

- *Typical situation:*
  - *we know:* the degrees $d(A)$ and $d(B)$ of expert confidence in statements A and B;
  - *we need:* to estimate the expert's degree of confidence in composite statements like $A\&B$, $A \vee B$:

$$d(A\&B) \approx f_\&(d(A), d(B));$$

$$d(A \vee B) \approx f_\vee(d(A), d(B));$$

$$d(\neg A) \approx f_\neg(d(A)).$$

- The functions providing such estimates are called *fuzzy logic operations*:
  - and-operations (a.k.a. t-norms),
  - or-operations (a.k.a. t-conorms),
  - negation operations, etc.

3. Background. II. Fuzzy Logic Operations (cont-d)

- Fuzzy logic operations must satisfy natural properties.
- Example 1:
  - *Fact:* $A\&B$ means the same as $B\&A$.
  - *Property:* the and-operation $f_\&(a,b)$ must be commutative:
  $$f_\&(a,b)=f_\&(b,a).$$
- Example 2:
  - *Fact:* $A\&(B\&C)$ means the same as $(A\&B)\&C$.
  - *Property:* the and-operation $f_\&(a,b)$ must be associative:
  $$f_\&(a,f_\&(b,c))=f_\&(f_\&(a,b),c).$$
- *Known:* there exist a complete descriptions of all the operations that satisfy such properties.

4. Formulation of the Problem

- *In principle:* we can have very *complex* fuzzy logic operations.
- *In practice:* mostly simple *algebraic* operations are used:
  - linear;
  - quadratic;
  - fractional-linear; etc.
- *Foundational challenge:* how do we classify such algebraic fuzzy operations?
- *What we prove in this talk:*
  - to classify *algebraic* fuzzy logic operations,
  - we do not need to use *all* the usual properties.

5. Motivating Result: Description of All Quadratic And-Operations

- Consider quadratic functions $f_\& : [0,1] \times [0,1] \to [0,1]$:

$$f_\&(a,b) = c_0 + c_1 \cdot a + c_b \cdot b + c_{aa} \cdot a^2 + c_{ab} \cdot a \cdot b + c_{bb} \cdot b^2.$$

- *Properties:*
  - the function $f_\&(a,b)$ is *monotonic* (*non-decreasing*) in each variable;
  - $f_\&$ is *conservative* in the sense that it coincides with the usual logical operation $a\&b$ for $a, b \in \{0,1\}$:

$$f_\&(0,0) = f_\&(0,1) = f_\&(1,0) = 0; \quad f_\&(1,1) = 1.$$

- *Result* (H.T. Nguyen, V. Kreinovich): the only quadratic and-operation with these properties is $f_\&(a,b) = a \cdot b$.
- *Comment:* we did not use commutativity or associativity.

6. Description of All Quadratic Or-Operations

- Consider quadratic functions $f_\vee : [0,1] \times [0,1] \to [0,1]$:

$$f_\vee(a,b) = c_0 + c_1 \cdot a + c_b \cdot b + c_{aa} \cdot a^2 + c_{ab} \cdot a \cdot b + c_{bb} \cdot b^2.$$

- *Properties:*
  - the function $f_\vee(a,b)$ is *monotonic* (*non-decreasing*) in each variable;
  - $f_\vee$ is *conservative* in the sense that it coincides with the usual logical operation $a \vee b$ for $a, b \in \{0,1\}$:

$$f_\vee(0,0) = 0, \quad f_\vee(0,1) = f_\vee(1,0) = f_\&(1,1) = 1.$$

- *Result:* the only quadratic and-operation with these properties is $f_\vee(a,b) = a + b - a \cdot b$.
- *Comment:* we did not use commutativity or associativity.

7. Negation Operations: Usual Properties

- *Main algebraic property:*
  - *Fact:* $\neg(\neg A)$ means the same as A.
  - *Property:* the negation operation $f_\neg(a)$ must satisfy the property:
$$f_\neg(f_\neg(a)) = a.$$
- *Monotonicity:* the more we believe in A, the less we believe in $\neg A$.
- *Conclusion:* the function $f_\neg(a)$ must be non-increasing.
- *Conservative:* for $a = 0$ ("false") and for $a = 1$ ("true"), $f_\neg(a)$ must coincide with the truth value of "not a":
$$f_\neg(0) = 1, \quad f_\neg(1) = 0.$$

8. Description of All Quadratic Negation Operations

- Consider quadratic functions $f_\neg : [0,1] \to [0,1]$:

$$f_\neg(a) = c_0 + c_1 \cdot a + c_{aa} \cdot a^2. \quad (1)$$

- *Properties:*
  - the function $f_\neg(a)$ satisfies the property

    $$f_\neg(f_\neg(a)) = a \text{ for all } a;$$

  - $f_\neg$ is *conservative* in the sense that it coincides with the usual logical operation $\neg a$ for $a \in \{0,1\}$:

    $$f_\neg(0) = 1, \quad f_\neg(1) = 0.$$

- *Result:* the only quadratic negation operation with these properties is $f_\neg(a) = 1 - a$.
- *Comment:* we did not use monotonicity.

9. Description of All Fractional-Linear Negation Operations

- Consider fractional-linear functions $f_\neg : [0,1] \to [0,1]$:

$$f_\neg(a) = \frac{a + b \cdot x}{c + d \cdot x}.$$

- *Properties:*
  - the function $f_\neg(a)$ satisfies the property

    $$f_\neg(f_\neg(a)) = a \text{ for all } a;$$

  - $f_\neg$ is *conservative* in the sense that it coincides with the usual logical operation $\neg a$ for $a \in \{0, 1\}$:

    $$f_\neg(0) = 1, \quad f_\neg(1) = 0.$$

- *Result:* the only fractional-linear negation operation with these properties is $f_\neg(a) = 1 - a$.
- *Comment:* we did not use monotonicity.

10. Proof of the Result about Quadratic Negation Operations

- *General formula:* $f_\neg(a) = c_0 + c_1 \cdot a + c_{aa} \cdot a^2$.
- The condition $f_\neg(0) = 1$ leads to $c_0 = 1$.
- Now, the condition $f_\neg(1) = 0$ leads to $c_{aa} = -1 - c_a$.
- Hence, $f_\neg(a) = 1 - a^2 + c_a \cdot (a - a^2)$.
- For this expression, the condition $f_\neg(f_\neg(a)) - a = 0$ takes the form:

$$(-1 - 2c_a - {c_a}^2) \cdot a + (2 + 3c_a - {c_a}^3) \cdot a^2 +$$

$$(c_a + 2{c_a}^2 + {c_a}^3) \cdot a^3 + (-1 - 3c_a - 3{c_a}^2 - {c_a}^3) \cdot a^4 = 0.$$

- *Comment:* we combined we combined terms corresponding to different powers of a.

11. Proof (cont-d)

- *Reminder:* for all a, we have

$$(-1 - 2c_a - {c_a}^2) \cdot a + (2 + 3c_a - {c_a}^3) \cdot a^2 +$$
$$(c_a + 2{c_a}^2 + {c_a}^3) \cdot a^3 + (-1 - 3c_a - 3{c_a}^2 - {c_a}^3) \cdot a^4 = 0.$$

- *Fact:* a polynomial is equal to zero only when all the coefficients are equal to zero.
- *Conclusion:* $-1 - 2c_a - {c_a}^2 = 0,\ \ 2 + 3c_a - {c_a}^3 = 0,$

$$c_a + 2{c_a}^2 + {c_a}^3 = 0, \quad -1 - 3c_a - 3{c_a}^2 - {c_a}^3 = 0.$$

- First equation means $-(1 + c_a)^2 = 0$, hence $1 + c_a = 0$ and $c_a = -1$.
- For $c_a = -1$, the formula $f_\neg(a) = 1 - a^2 + c_a \cdot (a - a^2)$ turns into $f_\neg(a) = 1 - a^2 - (a - a^2) = 1 - a$.
- So, $f_\neg(a) = 1 - a$ is the only quadratic negation operation. The result is proven.

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