

# For Describing Uncertainty, Ellipsoids Are Better than Generic Polyhedra and Probably Better than Boxes: A Remark

Olga Kosheleva and Vladik Kreinovich

University of Texas at El Paso  
El Paso, TX 79968, USA

olgak@utep.edu, vladik@utep.edu

Need for Describing . . .

Need for Describing . . .

Convex Set-Based . . .

Which Set  $S_0$  Should . . .

Which Set  $S_0$  Should . . .

Main Conclusion: . . .

Ellipsoids Are . . .

Ellipsoids Are . . .

Acknowledgments

Home Page

Title Page

⏪

⏩

◀

▶

Page 1 of 10

Go Back

Full Screen

Close

Quit

# 1. Need for Describing Sets of Possible Values

- Measurement and estimates are never 100% accurate.
- As a result, we usually do not know the exact value of a physical quantity.
- We usually know the set of possible values of this quantity.
- For a single quantity, this set is usually an interval.
- Representing an interval in a computer is easy: e.g., we can represent an interval by its endpoints.
- For several quantities  $x_1, \dots, x_n$ :
  - in addition to interval bounds on each of these quantities,
  - we often have additional restrictions on their combinations.

## 2. Need for Describing Sets of Possible Values (cont-d)

- In addition to interval bounds on quantities  $x_1, \dots, x_n$ , we often have restrictions on their combinations.
- As a result, the set of possible values of  $x = (x_1, \dots, x_n)$  can have different shapes.
- The space of all possible sets is infinite-dimensional.
- This means that we need infinitely many real-valued parameters to represent a generic set.
- In a computer, at any given time, we can only store finitely many parameters.
- So, we cannot represent generic sets exactly.
- We need to approximate them by sets from a finite-parametric family.

### 3. Convex Set-Based Representation of Sets

- When  $x_i$  are spatial coordinates, we can use a different coordinate system  $y_i = \sum_{j=1}^m t_{ij} \cdot x_j$ .
- In view of this, a reasonable way to select a finite-parametric set is:
  - to pick a bounded symmetric convex set  $S_0$  with non-empty interior, and
  - to use images  $TS_0$  of this set  $S_0$  under arbitrary linear transformations  $T$ .
- If we start with a Euclidean unit ball  $S_0 = B \stackrel{\text{def}}{=} \left\{ x : \sum_{i=1}^n x_i^2 \leq 1 \right\}$ , we get the family of ellipsoids.
- If we start with a unit cube  $C$ , we get the family of all boxes (plus the corresponding parallelepipeds).
- We can also pick a symmetric convex polyhedron  $P$ .

Need for Describing...

Need for Describing...

Convex Set-Based...

Which Set  $S_0$  Should...

Which Set  $S_0$  Should...

Main Conclusion:...

Ellipsoids Are...

Ellipsoids Are...

Acknowledgments

Home Page

Title Page

◀◀

▶▶

◀

▶

Page 4 of 10

Go Back

Full Screen

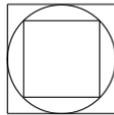
Close

Quit

## 4. Which Set $S_0$ Should We Choose?

- Once we pick a set  $S_0$ , we can (precisely) represent sets  $S$  of the type  $TS_0$ .
- If we start with such a set  $S$ , we enclose it into a set  $TS_0 = S$ .
- Then, we want to enclose  $TS_0$  in a set  $\lambda \cdot S$  corresponding to the original  $S$ -based representations.
- We get the same original set  $S = TS_0$  back, with  $\lambda = 1$ .
- For sets  $S$  which are different from  $TS_0$ , the  $S_0$ -based representation is only approximate.
- We start with a set  $S$ .
- We enclose it in a set  $TS_0 \supseteq S$  for an appropriate  $T$ .
- We then try to enclose  $TS_0$  in a set of the type  $\lambda \cdot S$ .
- Then we inevitably get  $\lambda > 1$ .

## 5. Which Set $S_0$ Should We Choose?



- We have  $S \subseteq TS_0 \subseteq \lambda \cdot S$ .
- The smaller  $\lambda$ , the better the approximation.
- As a measure  $d(S_0, S)$  of accuracy of approximating  $S$  by  $S_0$ , we use the smallest  $\lambda$ :

$$d(S_0, S) = \inf\{\lambda : \exists T (S \subseteq TS_0 \subseteq \lambda \cdot S)\}.$$

- This quantity is known as a *Banach-Mazur distance* between the convex sets  $S$  and  $S_0$ .
- As a measure of quality  $Q(S_0)$  of choosing  $S_0$ , we select the worst-case approximation accuracy

$$Q(S_0) \stackrel{\text{def}}{=} \sup_S d(S_0, S).$$

Need for Describing...

Need for Describing...

Convex Set-Based...

Which Set  $S_0$  Should...

Which Set  $S_0$  Should...

Main Conclusion:...

Ellipsoids Are...

Ellipsoids Are...

Acknowledgments

Home Page

Title Page



Page 6 of 10

Go Back

Full Screen

Close

Quit

## 6. Main Conclusion: Ellipsoids Are Better Than Generic Polyhedra

- *John's Theorem*: for the Euclidean unit ball  $B$ ,  $d(B, S) \leq \sqrt{n}$  for all symmetric convex sets  $S$ .
- Thus, we have  $Q(B) \leq \sqrt{n}$  (actually  $Q(B) = \sqrt{n}$ ).
- *Gluskin's Theorem* for polyhedra  $P, P'$ :

$$\exists c > 0 \forall n \exists P \exists P' (d(P, P') \geq c \cdot n).$$

- For this polyhedron  $P$ , we have  $d(P) \geq c \cdot n$ .
- Moreover, if we take a convex hull  $P$  of  $2n$  points randomly selected from a unit Euclidean sphere, then:

$$Q(P) \geq c \cdot n \text{ with high probability.}$$

- For large  $n$ ,  $c \cdot n \gg \sqrt{n}$  and thus,  $Q(B) \ll Q(P)$ .
- This shows that for large dimensions, ellipsoids are indeed better than generic polyhedra.

Need for Describing...

Need for Describing...

Convex Set-Based...

Which Set  $S_0$  Should...

Which Set  $S_0$  Should...

Main Conclusion:...

Ellipsoids Are...

Ellipsoids Are...

Acknowledgments

Home Page

Title Page



Page 7 of 10

Go Back

Full Screen

Close

Quit

## 7. Ellipsoids Are Probably Better Than Boxes

- A unit ball  $B$  and a unit cube  $C$  are unit balls

$$B_p \stackrel{\text{def}}{=} \{x : \|x\|_p \leq 1\} \text{ in the } \ell_p\text{-metric } \|x\|_p \stackrel{\text{def}}{=} \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} :$$

- $B$  is a unit ball in the  $\ell_2$ -metric:  $B = B_2$ ;
- $C$  is a unit ball in the  $\ell_\infty$ -metric:  $C = B_\infty$ .
- The exact values of  $d(B_p, B_q)$  are known only when both  $p$  and  $q$  are on the same side of 2.
- In this case,  $d(B_p, B_q) = n^{|1/p-1/q|}$ .
- *Example:* for  $p = 1$  and  $q = 2$ , we get  $d(B_1, B_2) = \sqrt{n}$ .
- These values have the property that when  $p < q$ , then  $d(B_p, B_q) \uparrow$  when  $p \downarrow$  or when  $q \uparrow$ .
- For  $n = 2$ ,  $B_1$  (rhombus) and  $B_\infty$  (square) are linearly equivalent.
- Thus,  $d(B_1, B_\infty) = 0 < d(B_2, B_\infty)$  (no monotonicity).

Need for Describing ...

Need for Describing ...

Convex Set-Based ...

Which Set  $S_0$  Should ...

Which Set  $S_0$  Should ...

Main Conclusion: ...

Ellipsoids Are ...

Ellipsoids Are ...

Acknowledgments

Home Page

Title Page



Page 8 of 10

Go Back

Full Screen

Close

Quit

## 8. Ellipsoids Are Probably Better Than Boxes (cont-d)

- For  $n = 2$ , we have an anomaly:  $B_1 = TB_\infty$ .
- As a result,  $d(B_p, B_q)$  is *not* monotonic in  $p$  and  $q$ .
- For  $n > 3$ , we do not have this anomaly.
- Therefore, it is reasonable to conjecture that for  $n > 3$ ,  $d(B_p, B_q)$  is monotonic in  $p$  and  $q$ .
- Under this hypothesis,  $d(B_\infty, B_1) > d(B_2, B_1) = \sqrt{n}$ .
- Thus,  $Q(B_\infty) \geq d(B_\infty, B_1) > \sqrt{n}$ .
- Since  $Q(B_2) = \sqrt{n}$ , we therefore conclude that

$$Q(B_2) < Q(B_\infty).$$

- Thus, ellipsoids are better than boxes.
- This is in line with a general result that, under certain conditions, ellipsoids are the best approximators.

Need for Describing ...

Need for Describing ...

Convex Set-Based ...

Which Set  $S_0$  Should ...

Which Set  $S_0$  Should ...

Main Conclusion: ...

Ellipsoids Are ...

Ellipsoids Are ...

Acknowledgments

Home Page

Title Page



Page 9 of 10

Go Back

Full Screen

Close

Quit

## 9. Acknowledgments

This work was supported in part:

- by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721,
- by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and
- by a grant on F-transforms from the Office of Naval Research.

*Need for Describing ...*

*Need for Describing ...*

*Convex Set-Based ...*

*Which Set  $S_0$  Should ...*

*Which Set  $S_0$  Should ...*

*Main Conclusion: ...*

*Ellipsoids Are ...*

*Ellipsoids Are ...*

*Acknowledgments*

*Home Page*

*Title Page*



*Page 10 of 10*

*Go Back*

*Full Screen*

*Close*

*Quit*