How to Make a Solution to a Territorial Dispute More Realistic: Taking into Account Uncertainty, Emotions, and Step-by-Step Approach

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1. Territorial Division: Formulation of the Problem

- In many real-life situations, there is a dispute over a territory:
  - from conflicts between neighboring farms
  - to conflict between states.
- As a result of a conflict, none of the sides can use this territory efficiently.
- In such situations, it is desirable to come up with a mutually beneficial agreement.
- The current solution is based on the work by the Nobelist J. Nash.
- Nash showed that the best mutually beneficial solution maximizes the product of all the utilities.

2. Nash's Solution: From a Theoretical Formulation to Practical Recommendations

- Let $u_i(x)$ be the utility (per area) of the i-th participant at location x.
- We should select a partition for which the product $\prod\limits_{i=1}^{n} U_i$ is the largest, where:
  - $U_i \stackrel{\text{def}}{=} \int_{S_i} u_i(x)\,dx$ and
  - S_i is the set allocated to the i-th participant.
- *Solution:* for some t_i, to assign each location x to the participant i with the largest ratio $u_i(x)/t_i$.
- The parameters t_i must be determined from the requirement that the $\prod\limits_{i=1}^{n} U_i \to \max$.
- For two participants, $x \in S_1$ if $\dfrac{u_1(x)}{u_2(x)} \geq t \stackrel{\text{def}}{=} \dfrac{t_1}{t_2}$.

3. Nash's Solution: Advantages and Limitations

- Nash's solution is in perfect agreement with common sense description as formalized by fuzzy logic:
  - we want he first participant to be happy *and* the second participant to be happy, etc.
  - the degree of happiness of each participant can be described by his or her utility;
  - to represent "and", it's reasonable to use one of the most frequently used fuzzy "and"-operations $a \cdot b$.
- Nash's solution assumes that we know the exact values $u_i(x)$.
- In reality, we know the values $u_i(x)$ only approximately.
- For example, we only know the interval $[\underline{u}_i(x), \overline{u}_i(x)]$ containing $u_i(x)$.
- How can we take this uncertainty into account?

4. Nash's Solution: Limitations (cont-d)

- The above solution assumes that all the sides are making their decisions on a purely rational basis.
- In reality, emotions are often involved.
- How can we take these emotions into account?
- Finally, the above formula proposes an immediate solution.
- But participants often follow step-by-step approach:
  - they first divide a small part,
  - then another part, etc.
- This also needs to be taken into account.
- In this talk, we show how to take all this into account.

5. How to Take Uncertainty into Account

- *Reminder:* we often only know the bounds on $u_i(x)$: $\underline{u}_i(x) \le u_i(x) \le \overline{u}_i(x)$.
- In this case, for each allocation S_i, we only know the *interval* $[\underline{U}_i, \overline{U}_i]$ of possible values of utility:

$$\underline{U}_i \stackrel{\text{def}}{=} \int_{S_i} \underline{u}_i(x)\, dx; \quad \overline{U}_i \stackrel{\text{def}}{=} \int_{S_i} \overline{u}_i(x)\, dx$$

- In situations with interval uncertainty, decision theory recommends using $\widetilde{U}_i = \alpha_i \cdot \overline{U}_i + (1 - \alpha_i) \cdot \underline{U}_i$
  - $\alpha_i \in [0, 1]$ be i-th participant's degree of optimism
  - Similarly, we can use $\widetilde{U}_i = \int_{S_i} \widetilde{u}_i\, dx$, where $\widetilde{u}_i(x) \stackrel{\text{def}}{=} \alpha_i \cdot \overline{u}_i(x) + (1 - \alpha_i) \cdot \underline{u}_i(x)$
- We acquire the same formulation, so, we assign each location x to a participant with the largest ratio $\widetilde{u}_i(x)/t_i$.

6. Example

- Let us assume that different participants assign the same utility to all the locations:

$$\underline{u}_i(x) = \underline{u}_j(x) \text{ and } \overline{u}_i(x) = \overline{u}_j(x) \text{ for all } i \text{ and } j.$$

- The only difference between the participants is that they have different optimism degrees $\alpha_i \neq \alpha_j$.
- Without losing generality, let $\alpha_i > \alpha_j$.
- Then, the above optimization implies that a point is allocated to i-th zone if $\dfrac{\overline{u}(x) - \underline{u}(x)}{\underline{u}(x)} \geq t$; so:
  - a more optimistic participant gets the locations with higher uncertainty, while
  - a more pessimistic one get locations with lower uncertainty.

7. How to Take Emotions Into Account

- Emotions mean that instead of maximizing U_i, participants maximize $U_i^{\rm emo} = U_i + \sum\limits_j \alpha_{ij} \cdot U_j$.
- Here, α_{ij} describes the feelings of the i-th participant towards the j-th one:
  - $\alpha_{ij} > 0$ indicate positive feelings;
  - $\alpha_{ij} < 0$ indicate negative feelings;
  - $\alpha_{ij} = 0$ indicate indifference.
- Nash's solution is to maximize the product $\prod\limits_i U_i^{\rm emo}$.
- *Result:* for some t_i we assign each location x to a participant with the largest ratio $\widetilde{u}_i(x)/t_i$.
- *Main difference:* the thresholds t_i change.
- A participant with $\alpha_{ij} > 0$ gets fewer locations x, since his utility is improved via happiness of others.

8. What If Emotions Are Negative?

- When emotions are negative, i.e., when $\alpha_{ij} < 0$, then, somewhat surprisingly, we get a positive effect.
- Specifically, negative emotions stimulate equality.
- Indeed, all the sides agree to a division only if their utilities $U_i^{\rm emo}$ are non-negative.
- For example, when $\alpha_{12} = \alpha_{21} = -1$, then:
  - the only way to guarantee that both values $U_1^{\rm emo} = U_1 - U_2$ and $U_2^{\rm emo} = U_2 - U_1$ are non-negative is
  - when the values U_1 and U_2 are equal to each other.
- For other values α_{ij}:
  - we do not get $U_i = U_j$, but
  - we get bounds limiting how much U_i and U_j can differ from each other: $0 < c \le \dfrac{U_i}{U_j} \le C$.

9. Immediate Solution vs. Step-by-Step Approach

- It is desirable to arrive at an immediate solution, but in international affairs, this is not common.
- So, we approach the problem in a location-by-location basis.
- It turns out that the resulting arrangement is not optimal.
- In small vicinities of each location x, utility functions $u_i(x)$ do not change much.
- So, we can safely assume that in the vicinity, each utility function is a constant $u_i(x) = u_i$.
- Thus, the utility $U_i = \int_{S_i} u_i(x)\,dx$ is proportional to the area A_i of the set S_i: $U_i = u_i \cdot A_i$.
- Then, the optimal division means selecting A_i for which $\sum\limits_{i=1}^n A_i = A$ and $\prod\limits_{i=1}^n U_i = \prod\limits_{i=1}^n (u_i \cdot A_i) \to \max$.

10. Immediate Solution vs. Step-by-Step (cont-d)

- The optimal division means selecting A_i that maximize $\prod_{i=1}^{n} U_i = \prod_{i=1}^{n} (u_i \cdot A_i) \to \max$ under the constraint

$$\sum_{i=1}^{n} A_i = A.$$

- Solution is $A_i = \dfrac{A}{n}$: each vicinity is divided equally.
- Let us show that this is not optimal.

11. Step-by-Step Approach: An Example

- An area S consists of two equal parts:
  - the first part is useless for the 1st participant, but valuable to the second one
  - the second part is valuable for the first participant, but useless for the second one
- A clear optimal solution is to allocate:
  - the first part to the second participant and
  - the second part to the first participant.
- In a step-by-step solution, we divide each part equally.
- As a result, each participant gets only half of the area which is useful to this participant (non-optimal)
- Recommendation: try to solve the problem as a whole, and avoid step-by-step solutions.

12. Auxiliary Question: Should we Divide in the First Place?

- At first glance, it may seem that:
  - instead of dividing a disputed territory,
  - it is desirable to show a brotherly/sisterly spirit and control it jointly.
- This may work at times.
- However, as we show, in general, this strategy will lead to a suboptimal solution: in almost all cases,
  - the product of utilities is the largest when we divide
  - and not when we share the control.

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14. How to Take Emotions Into Account: Proof

- Let us assume that in the optimal division, location x_0 is allocated to the i_0-th participant.
- This means that:
  - if re-allocate a small neighborhood of x_0 (of area δ) to participant j_0,
  - then the product $\prod\limits_i U_i^{\rm emo}$ will decrease;
  - so its logarithm $L = \ln\left(\prod\limits_i U_i^{\rm emo}\right)$ also decreases.
- Here, $U_{i_0} = \int_{S_{i_0}} u_{i_0}(x)\,dx$ decreases by

$$\Delta U_{i_0} = -u_{i_0}(x_0)\cdot\delta.$$

- $U_{j_0} = \int_{S_{j_0}} u_{j_0}(x)\,dx$ increases by $\Delta U_{j_0} = u_{j_0}(x_0)\cdot\delta$.
- All other U_i remain unchanged: $\Delta U_i = 0$ for $i \ne i_0, j_0$.

15. Emotions: Proof (cont-d)

- Thus, for the changes ΔU_i^{emo}:
  - for $i = i_0$, we have $\Delta U_{i_0}^{\text{emo}} = \Delta U_{i_0} + \alpha_{i_0 j_0} \cdot \Delta U_{j_0}$;
  - for $i = j_0$, we have $\Delta U_{j_0}^{\text{emo}} = \Delta U_{i_0} + \alpha_{j_0 i_0} \cdot \Delta U_{i_0}$;
  - for all other i, $\Delta U_i^{\text{emo}} = \alpha_{i i_0} \cdot \Delta U_{i_0} + \alpha_{i j_0} \cdot \Delta U_{j_0}$.
- For $L = \sum\limits_i \ln(U_i^{\text{emo}})$, we have $\Delta L = \sum\limits_i \dfrac{\Delta U_i^{\text{emo}}}{U_i^{\text{emo}}}$.
- So $\Delta L \le 0$ takes the form $a \cdot u_{i_0}(x_0) + b \cdot u_{j_0}(x_0) \le 0$, or, equivalently, $\dfrac{u_{i_0}(x_0)}{u_{j_0}(x_0)} \ge c$.
- If x_0 was originally allocated to j_0, we get same inequality with $-\delta$ instead of δ, so $\dfrac{u_{i_0}(x_0)}{u_{j_0}(x_0)} \le c$.
- Thus, in the optimal partition, each participant i indeed gets all locations for which $u_i(t)/t_i$ is the largest.