From Fuzzy Universal Approximation to Fuzzy Universal Representation: It All Depends on the Continuum Hypothesis

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1. Need to Translate Expert Statements into Precise Terms

- In many practical situations,
  - there is a correlation between two quantities x and y,
  - the only information that we have to describe this correlation are expert statements, and
  - these statement use imprecise ("fuzzy") natural language words, such as "small".
- For example, an expert can say that if x is small, then y is big, and vice versa.

2. Fuzzy Logic Provides the Desired Translation

- Fuzzy logic is a technique that translates this knowledge into precise mathematical terms.
- In this technique, each fuzzy term A is described by a function $A(x)$ assigning,
  - to each possible value x of the corresponding quantity,
  - a degree $A(x)$ to which this value has the appropriate property (e.g., is small).
- The expert gives us rules $A_i(x) \Rightarrow B_i(y)$.
- The degree $d(x, y)$ to which each pair (x, y) is possible can be described as the degree to which:
  - either the first rule is satisfied (i.e., $A_1(x)$ is true and $B_1(y)$ is true)
  - or the second rule is satisfied, etc.

3. Fuzzy Logic (cont-d)

- The degree $d(x, y)$ to which each pair (x, y) is possible can be described as the degree to which:
  - either the first rule is satisfied (i.e., $A_1(x)$ is true and $B_1(y)$ is true)
  - or the second rule is satisfied, etc.
- One possible way to interpret "and" is to use product, and "or" is sum.
- Then, the desired degree takes the form

$$d(x, y) = \sum_{i=1}^{n} A_i(x) \cdot B_i(y).$$

4. Formulation of the Problem

- Fuzzy rules lead to the formula
$$d(x,y)=\sum_{i=1}^{n} A_i(x)\cdot B_i(y).$$
- It is known that the above expression has a universal approximation property:
  - for every $\varepsilon>0$,
  - every continuous function on a box can be ε-approximated by such sums.
- *Natural question:* when can we get an *exact* representation of every function?

5. Let us Formulate the Problem in Precise Terms: First Attempt

- We start with the formula

$$d(x, y) = \sum_{i=1}^{n} A_i(x) \cdot B_i(y).$$

- The simplest way to interpret the above question is to ask whether there exists an integer n such that
  - *any* function of two variables can be represented in this form with this particular n.
- It turns our that this is not possible: this was proven by Roy O. Davies from Purdue in 1974.

6. Let us Formulate the Problem in Precise Terms: Second Attempt

- We cannot have universal representation by using a fixed finite number of terms.
- A natural next idea is to have a representation in which:
  - the number of terms is finite for every function $d(x, y)$ and for every pair (x, y), but
  - this number of terms may be different for different functions $d(x, y)$ and different pairs (x, y).
- Thus, we arrive at the following definition.

7. Definition and the Resulting Question

- **Definition.** *We say that there is a* universal representation property *if:*
  - *every function* $d(x, y)$ *of two variables*
  - *can be represented as the sum*

$$d(x,y) = \sum_{i=1}^{\infty} A_i(x) \cdot B_i(y),$$

  - *so that for every pair* (x, y)*, only finitely many terms in the sum are different from 0.*

- *Resulting question:* is there a universal representation property?

8. Somewhat Surprising Answer: It All Depends on the Continuum Hypothesis

- Intuitively, we expect:
  - either a positive answer to the above question – i.e., a proof that the universal representation is possible,
  - or a proof that such a universal representation is not possible.
- The actual answer is not what we would intuitively expect.
- **Proposition 1.** *The universal representation property is equivalent to the Continuum Hypothesis.*
- This was also proved by Roy O. Davies in 1974.
- So, the universal representation property depends on a somewhat obscure hypothesis from set theory.

9. What is the Continuum Hypothesis: Reminder

- In the late 19th century, Georg Cantor invented set theory.
- Set theory is now the foundations of mathematics.
- He proved that each infinite subset S of the set N of natural numbers is equivalent to N – in the sense that:
  - there is a 1-to-1 correspondence
  - between N and S.
- He also proved that the *continuum* – i.e., the set R of real numbers – is *not* equivalent to N.
- Cantor conjectured that every infinite subset S of the continuum R is not equivalent either to N or to R.
- This conjecture became known as *Continuum Hypothesis* (CH).

10. Continuum Hypothesis (cont-d)

- Working mathematicians usually assume this hypothesis.
- But can this hypothesis can be proven or disproven based on other – more intuitive – axioms?
- This remained an open problem for a long time.
- The first breakthrough came from the famous logician Kurt Gödel.
- Gödel proved, in 1940, that the negation of CH cannot be proven in set theory.
- He proved it by showing that:
  - if set theory is consistent, i.e., has a model,
  - then, based on this model, we can build another model in which CH is true.

11. Continuum Hypothesis (final)

- Can CH or its negation be derived from other axioms of set theory?
- This question was settled in the 1960s, when Paul Cohen proved that CH is independent of set theory.
- In other words, we can neither prove not disprove it based on other axioms of set theory.
- For this result, Paul Cohen was awarded the Fields Medal – the math. equivalent of the Nobel Prize.

12. Why This Result is Interesting

- At first glance:
  - the Continuum Hypothesis is an obscure statement of set theory,
  - of little interest to working mathematicians
  - and probably of even less interest to applications of mathematics.
- However, surprisingly, this abstract statement is equivalent to something much more practical.
- Namely, CH is equivalent the universal representation property for fuzzy systems.

13. Why This Result is Interesting (cont-d)

- Of course, one can argue that
  - in practice, when everything is measured and implemented with some accuracy anyway,
  - all we care about is the universal *approximation* property.
- However, the universal representation property also makes application sense.
- This property shows that we can have an approximation in which:
  - for every property $d(x, y)$ and for every pair (x, y),
  - the number of non-zero terms remains constant no matter how much accuracy we seek;
  - thus, the number of expert rules remains constant when we increase the accuracy.

14. References

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