Decision Making Under General Set Uncertainty: Additivity Approach

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1. Need for Decision Making Under Interval Uncertainty

- In many practical situations, we do not know the exact consequences of different alternatives; for example:
  - we may know that investing $1000 in a project will bring us between $10 and $40 in a year,
  - but we do not know how much exactly.
- On the other hand, there are usually some alternatives with known results.
- E.g., we can place this amount into a saving account at the bank.
- This will bring us exactly $20 at the end of the year.
  - In the first case, all we know about our gain is it is somewhere in the interval $[10, 40]$.
  - In the second case the gain is 20.

2. Need for Decision Making (cont-d)

- Which of these two alternatives is better?
- To be able to make a choice, we must be able to compare intervals with real numbers and with intervals.

3. From Interval to Set Uncertainty

- In some cases, we know that not all the values from the corresponding interval are possible.
- For example, we may know that we will either get \$10 or \$40.
- In this case, the set of the possible values is not the whole interval $[10, 40]$, but the 2-point set $\{10, 40\}$.
- We may have more complicated situations.
- For example, we may have either \$10, or some value between \$30 and \$40.
- In this case, the set of possible values is $\{10\} \cup [30, 40]$.
- To make decisions in such situations, we need to compare sets with intervals, numbers, and other sets.

4. Additivity: the Main Idea Behind such Decision Making

- Suppose that:
  - in one situation, we have a set S_1 of possible gains s_1, and
  - in another independent situation, we have a set S_2 of possible gains s_2.
- Then, by participating in both situation, we can gain the value $s = s_1 + s_2$.
- The set S of possible values of the overall gain can be obtained if we consider all possible $s_1 \in S_1$ and $s_2 \in S_2$:

$$S = S_1 + S_2 \stackrel{\text{def}}{=} \{s_1 + s_2 : s_1 \in S_1 \text{ and } s_2 \in S_2\}.$$

- It is reasonable to assign, to each set S, the price $u(S)$ we can pay to participate in this situation.

5. Additivity (cont-d)

- If the sets S_1, S_2 have the same price $(u(S_1) = u(S_2))$, we say that these two sets are *equivalent*: $S_1 \equiv S_2$.
- The price to participate in both events should be equal to the sum of the prices: $u(S_1 + S_2) = u(S_1) + u(S_2)$.
- This property is known as *additivity*.
- Let $\mathcal{S}$ be a class of sets which is closed under addition.
- An equivalence relation $\equiv$ is called *additive* if:

$$\text{if } S_1 + S_2 = S_1' + S_2 \text{ then } S_1 \equiv S_1'.$$

- *For every additive function* u, *the relation* $S_1 \equiv S_2 \stackrel{\text{def}}{=} (u(S_1) = u(S_2))$ *is additive.*
- Indeed, if $S_1 + S_2 = S_1' + S_2$, then, due to additivity, $u(S_1) + u(S_2) = u(S_1') + u(S_2)$.
- Thus, $u(S_1') = u(S_1)$ and $S_1' \equiv S_1$.

6. Decision Making Under Interval Uncertainty: What Is Known

- In case the set of possible gains is an interval $[\underline{a}, \overline{a}]$, no matter what happens, we will get $\geq \underline{a}$ and $\leq \overline{a}$.
- Thus, the price of this interval cannot be lower than $\underline{a}$ and cannot be higher than $\overline{a}$.
- We say that a real-valued function u defined on the set of all intervals is *consistent* if for each interval, we have

$$\underline{a} \leq u([\underline{a}, \overline{a}]) \leq \overline{a}.$$

- *Every consistent additive function u on the set of all intervals has the form*

$$u([\underline{a}, \overline{a}]) = \alpha \cdot \overline{u} + (1 - \alpha) \cdot \underline{u}, \quad \text{for some } \alpha \in [0, 1].$$

7. Hurwicz Criterion

- This formula was first proposed by the future Nobel prize winner Leo Hurwicz.
- It is known as *Hurwicz optimism-pessimism criterion*.
- Optimism corresponds to $\alpha = 1$, when a decision maker values the interval as much as its largest value.
- So, in effect, he/she considers only the best value from this interval to be possible.
- Pessimism corresponds to $\alpha = 0$, when a decision maker values the interval as much as its smallest value.

8. Decision Making Under Set Uncertainty: What Is Known

- It is known how to make a decision when S is bounded and closed (i.e., contains all its limit points).
- *For every additive equivalence relation on the class of all bounded closed sets,* $S \equiv [\inf(S), \sup(S)]$.
- Thus, the utility of each set S is equal to the utility of the corresponding interval $[\inf(S), \sup(S)]$.

9. Proof of This Result

- Every bounded closed sets contains its limit points; in particular, it contains the points $\inf(S)$ and $\sup(S)$.
- Thus, $\{\inf(S), \sup(S)\} \subseteq S \subseteq [\inf(S), \sup(S)]$.
- So, by a clear set-inclusion monotonicity of set addition, we conclude that

$$\{\inf(S), \sup(S)\} + [\inf(S), \sup(S)] \subseteq S + [\inf(S), \sup(S)] \subseteq [\inf(S), \sup(S)] + [\inf(S), \sup(S)].$$

- However, one can easily check that

$$\{\inf(S), \sup(S)\} + [\inf(S), \sup(S)] = [\inf(S), \sup(S)] + [\inf(S), \sup(S)] = [2\inf(S), 2\sup(S)].$$

10. Proof of This Result (cont-d)

- Thus, the intermediate set $S+[\inf(S),\sup(S)]$ should be equal to the same interval:

$$S+[\inf(S),\sup(S)] = [\inf(S),\sup(S)]+[\inf(S),\sup(S)] = [2\inf(S),2\sup(S)].$$

- Since the equivalence relation is assumed to be additive, we conclude that $S\equiv[\inf(S),\sup(S)]$.
- The proposition is proven.

11. Remaining Problem

- Boundedness is reasonable: in all real-life situations, we have lower and upper bounds on possible gains:
  - in usual investments, we do not expect to gain millions, and
  - we do not exact to lose millions – since usually, we just do not have these millions to lose.
- However, the requirement that the set be closed may be too restrictive; for example:
  - we may know that the gain will be between 0 and \$100,
  - but we are sure that the gain cannot be zero and cannot be exactly \$100.

12. Remaining Problem (cont-d)

- In this case, the set S of possible values of gain is an open interval $(0, 100)$.
- This interval that does not contain its limit points 0 and 100.
- How can we make decision under such general (not necessarily closed) set uncertainty?
- This is a question that we analyze in this talk.

13. Main Result

- *For every additive equivalence relation on the class of all bounded sets,* $S \equiv [\inf(S), \sup(S)]$.
- In other words, not only every bounded closed set is equivalent to the corresponding interval.
- Every bounded set S (not necessarily closed one) is equivalent to the interval $[\inf(S), \sup(S)]$.

14. Proof

- Let us first show that each open or semi-open interval is equivalent to the corresponding closed interval.
- Indeed, one can easily check that

$$(\underline{a}, \overline{a}) + (\underline{a}, \overline{a}) = [\underline{a}, \overline{a}] + (\underline{a}, \overline{a}) = (2\underline{a}, 2\overline{a}).$$

- Thus, by definition of additivity of an equivalence relation, we get $(\underline{a}, \overline{a}) \equiv [\underline{a}, \overline{a}]$.
- Similarly, we have

$$(\underline{a}, \overline{a}] + (\underline{a}, \overline{a}) = [\underline{a}, \overline{a}] + (\underline{a}, \overline{a}) = (2\underline{a}, 2\overline{a}).$$

- So, we can conclude that $(\underline{a}, \overline{a}] \equiv [\underline{a}, \overline{a}]$.
- Also, one can check that:

$$[\underline{a}, \overline{a}) + (\underline{a}, \overline{a}) = [\underline{a}, \overline{a}] + (\underline{a}, \overline{a}) = (2\underline{a}, 2\overline{a}).$$

- Thus, $[\underline{a}, \overline{a}) \equiv [\underline{a}, \overline{a}]$.

15. Proof (cont-d)

- Let us now consider a general bounded set S.
- If this set contains both points $\inf(S)$ and $\sup(S)$, then $S \equiv [\inf(S), \sup(S)]$ from the proof of the previous Proposition.
- Thus, to complete our proof, it is sufficient to consider the case when either $\inf(S) \notin S$ or $\sup(S) \notin S$.
- Without losing generality, let us consider the case when

$$\inf(S) \notin S.$$

- Let us prove that in this case, we have

$$S + (\inf(S), \sup(S)) = (2\inf(S), 2\sup(S)).$$

- It is easy to check that

$$(\inf(S), \sup(S)) + (\inf(S), \sup(S)) = (2\inf(S), 2\sup(S)).$$

16. Proof (cont-d)

- Thus, the desired equality would imply that

$$S+(\inf(S),\sup(S)) = (\inf(S),\sup(S))+(\inf(S),\sup(S)).$$

- Thus, by additivity of the equivalence relation,

$$S \equiv (\inf(S),\sup(S)).$$

- Since we have shown that $(\inf(S),\sup(S)) \equiv [\inf(S),\sup(S)]$, we will thus be able to conclude that $S \equiv [\inf(S),\sup(S)]$.
- So, all we need to do is prove that

$$S+(\inf(S),\sup(S)) = (2\inf(S), 2\sup(S)).$$

- The two sets are equal if the first is contained in the second one, and vice versa.

17. Proof (cont-d)

- Here, $S \subseteq (\inf(S), \sup(S)]$, thus

$$S + (\inf(S), \sup(S)) \subseteq (\inf(S), \sup(S)] + (\inf(S), \sup(S)) = (2\inf(S), 2\sup(S)).$$

- Thus, to complete the proof, it is sufficient to prove that every $s \in (2\inf(S), 2\sup(S))$ is in $S + (\inf(S), \sup(S))$.
- This means this number s can be represented as $s_1 + s_2$, where $s_1 \in S$ and $s_2 \in (\inf(S), \sup(S))$.
- To prove this, let us consider two possible cases:

$$s \le \inf(S) + \sup(S) \text{ and } \inf(S) + \sup(S) < s.$$

- Let us first consider the case when $s \le \inf(S) + \sup(S)$.
- Since s is in the open interval $(2\inf(S), 2\sup(S))$, we have $2\inf(S) < s \le \inf(S) + \sup(S)$.

18. Proof (cont-d)

- In this case, for $s' \stackrel{\text{def}}{=} s - \inf(S)$, we get the inequality $\inf(S) < s' \le \sup(S)$.
- By definition of $\inf(S)$, for every $s' > \inf(S)$, there exists a point $s_1 \in S$ for which $s_1 < s'$.
- So, we have $\inf(S) < s_1 < s - \inf(S)$.
- The first inequality is strict since $s_1 \in S$ and we consider the case when $\inf(S) \notin S$.
- From the inequality $s_1 < s - \inf(S)$, we conclude that $\inf(S) < s_2 \stackrel{\text{def}}{=} s - s_1$.
- On the other hand, from the inequalities $s \le \inf(S) + \sup(S)$ and $\inf(S) < s_1$, we conclude that

$$s_2 = s - s_1 < (\inf(S) + \sup(S)) - \inf(S) = \sup(S).$$

- So, $s_2 \in (\inf(S), \sup(S))$.

19. Proof (cont-d)

- Thus, $s = s_1 + s_2$, where $s_1 \in S$ and $s_2 \in (\inf(S), \sup(S))$.
- Let us now consider the case when $\inf(S) + \sup(S) < s$, i.e., when $\inf(S) + \sup(S) < s < 2\sup(S)$.
- From this inequality, it follows that

$$\inf(S) < s - \sup(S) < \sup(S).$$

- By definition of $\sup(S)$:
  - for each value smaller than $\sup(S)$,
  - in particular, for the value $s - \sup(S)$,
  - there exists a larger value from the set S.
- Let us denote this larger value by s_1: $s - \sup(S) < s_1$.
- Thus, for $s_2 \stackrel{\text{def}}{=} s - s_1$, we get $s_2 < \sup(S)$.

20. Proof (cont-d)

- On the other hand, from $\inf(S) + \sup(S) < s$ and $s_1 \le \sup(S)$, it follows that

$$(\inf(S) + \sup(S)) - \sup(S) = \inf(S) < s_2 = s - s_1.$$

- So, $s_2 \in (\inf(S), \sup(S))$.
- Thus, $s = s_1 + s_2$, where $s_1 \in S$ and $s_2 \in (\inf(S), \sup(S))$.
- The proposition is proven.

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