Perfect Reproducibility Is Not Always Algorithmically Possible: A Pedagogical Observation

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1. Users Often Want Perfect Reproducibility

- Users of software and, more generally, users of computer-based systems often want perfect *reproducibility*: that
  - if we place the system in the exact same situation,
  - it should react the exact same way.
- Of course, if a real-life system includes sensors and measurements, we cannot have exact reproducibility.
- If we measure the same value several times, we may get different results.
- As a result, e.g., when we have a computer-controlled thermoregulation system, then:
  - even for the exact same temperature,
  - the sensors readings will be slightly different and thus, the system's reaction may be different.

2. At Least the Users Want Perfect Reproducibility in the Ideal Situation

- The above measurement uncertainty is well known.
- So what the users want is that:
  - the system's behavior be perfectly reproducible in the idealized situation,
  - when we can measure each quantity with any given accuracy.
- We provide simple arguments that even in this idealized case, perfect reproducibility is not always possible.

3. Description of the Simple Case Study

- Let us consider a very simple control situation, when we want to keep some quantity q at a given level q_0.
- To perform this task, we measure q.
- We consider an idealized case when:
  - for every integer n,
  - we can measure q with accuracy of n binary digits (i.e., with an accuracy 2^{-n}).
- In such a measurement, we get a measurement result q_n which is 2^{-n}-close to q: $|q - q_n| \le 2^{-n}$.
- Let us also consider a very simplified version of a controller, with only two options:
  - we can switch on a device that increases q,
  - or we can switch on a device that decreases q,
  - or we can do nothing.

4. Simple Case Study (cont-d)

- Example: regulating room temperature:
  - if the temperature is above a certain threshold, switch on the air conditioner,
  - if the temperature is below a certain threshold, switch on the heater, and
  - if the temperature is comfortable, do nothing.
- Keeping a satellite at a given height above earth:
  - if the height decreases, we switch on an engine that pushes the orbit up;
  - if the height increases, we switch on another engine that pushes the orbit down; and
  - if the height is close to desired one, do nothing.

5. What Control Strategy We Can Apply

- We would like to design a computer-based control system for this setting.
- This system can start by measuring the value of the desired quantity q with some initial accuracy of n_0 binary digits.
- Based on the result q_{n_0} of this measurement, we can make four possible decisions:
  - we can switch on the device that increases q; we will denote the corresponding decision by $+$;
  - we can switch on the device that decreases q; we will denote the corresponding decision by $-$;
  - we can decide to do nothing at this point; we will denote the corresponding decision by 0; or
  - we can select to perform a more accurate measurement.

6. Control Strategy (cont-d)

- In the last case:
  - the system will generate an integer $n > n_0$,
  - perform the measurement with accuracy 2^{-n},
  - based on the new measurement result q_n, again select one of these four options.
- After one or several iterations, we produce:
  - a plus decision $+$ (increase q),
  - a minus decision $-$ (decrease q), or
  - a 0 decision (do nothing).
- When the value q is sufficiently large ($q \ge \overline{q}$ for some $\overline{q} > q_0$), we should make a minus decision.
- When the value q is sufficiently small ($q < \underline{q}$ for some $\underline{q} < q_0$), we should make a plus decision.

7. Comment

- In real life, we often have the option of performing a more accurate measurement; for example:
  - if a person has fallen down and hurt himself, and an X-ray picture is inconclusive,
  - a doctor may order an MRI image to get a more accurate picture of the damage.
- The main difference between such real-life situations and our idealized situation is that:
  - in real life, there is always a limit of how accurately we can measure, while
  - in our idealized setting, we assume that we can perform the measurement with an arbitrary accuracy.

8. Can We Achieve Perfect Reproducibility in Such a Situation?

- Is it possible, in such an idealized situation, to achieve perfect reproducibility?
- In other words, is it possible to design a control strategy in such a way that:
  - for the same actual value of the parameter q,
  - the system would make the exact same decision when this value is encountered the next time?
- We prove that such a perfectly reproducible control algorithm is impossible.

9. Proof

- We will prove this impossibility by contradiction.
- Let us assume that such a perfectly reproducible control strategy *is* possible.
- Then, for each actual value q of the corresponding quantity, this control algorithm returns $+$, $-$, or 0.
- For values $q \geq \overline{q}$, all the decisions are minus decisions.
- Thus, the set S_- of all the values q for which the algorithm produces a minus recommendation is non-empty.
- For $q \leq \underline{q}$, all the decisions are $+$ recommendations.
- Thus, for these values q, we never make a minus decision.
- So, S_- only contains values which are larger than $\underline{q}$.
- Therefore, the set S_- is bounded from below.

10. Proof (cont-d)

- The set S_- is non-empty and bounded from below.
- Thus, this set has the greatest lower bound (infimum) $s \stackrel{\text{def}}{=} \inf(S_-)$.
- One can see that for each n, in the 2^{-n}-vicinity of the value s, there exist:
  - a point s_n^- for which the algorithm does not produce minus, and
  - a point s_n^+ for which the algorithm does produce minus.
- As s_n^-, we can simply take $s_n^- = s - 2^{-n}$.
- Since $s_n^- < s$, and s is the infimum of S_-, the system cannot return minus for the value s_n^-.
- The existence of the value $s_n^+ \in S_-$ for which $s_n^+ \le s + 2^{-n}$ is also easy to show.

11. Proof (cont-d)

- If there was no such s_n^+, this would mean that all the values from S_- are larger than $s + 2^{-n}$.
- Therefore, $s + 2^{-n}$ would be a lower bound for all the points from the set S_-.
- However, we know that s is the greatest lower bound.
- So the value $s + 2^{-n}$ which is larger than s cannot be a lower bound.
- Now, let us analyze what exactly our algorithm is supposed to return when $q = s$.
- The system's recommendation is based on the latest result of measuring q.
- This measurement result q_N is accurate only with accuracy 2^{-N} for some N, i.e., $|q_N - s| \le 2^{-N}$.

12. Proof (cont-d)

- Perfect reproducibility means that for the value s:
  - no matter the measurement result q_N is,
  - we should make the same recommendation.
- Thus, we should produce the same recommendation for all measurement results $q_N \in [s - 2^{-N}, s + 2^{-N}]$.
- In situations in which the actual value is $s_N^- = s - 2^{-N}$, one of the possible measurement results is $q_N = s_N^-$.
- Since this measurement result q_N is in the above interval $[s - 2^{-N}, s + 2^{-N}]$, this means that:
  - based on this measurement result,
  - we should make the same recommendation as for the value s.

13. Proof (cont-d)

- We know that for the value s_N^-, the recommendation is *not* minus, it is ether $+$ or 0.
- Thus, for the value s, we should produce the same recommendation of $+$ or 0.
- On the other hand:
  - in situations in which the actual value is
  $$s_N^+ \in [s, s + 2^{-N}],$$
  - also one of the possible measurement results is this same value $q_N = s_N^+$.
- Since $q_N \in [s - 2^{-N}, s + 2^{-N}]$, this means that:
  - based on this measurement result,
  - we should make the same recommendation as for the value s.

14. Proof (cont-d)

- We know that for the value s_N^+, the recommendation *is* minus.
- Thus, for the value s, we should produce the same recommendation minus.
- So, we get a contradiction:
  - on the other hand, for the value s, the system should issue the recommendation of minus or 0;
  - on the other hand, for the same value s, the system should issue the recommendation $+$.
- This contradiction shows that a perfectly reproducible control strategy is indeed not possible.

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