How to Explain Empirical Efficiency of Non-Associative Aggregation Operations?

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1. Need to describe experts' uncertainty in computer-based systems

- A significant part of our knowledge comes from experts.
- Experts are often not absolutely confident about their statements.
- For example, a medical doctor can say that certain of an X-ray images indicate possible cancer.
- However, the doctor may add that there are cases when other diseases show similar pattern.
- So, additional tests are needed before we start a treatment.
- To adequately represent the expert's knowledge in a computer-based system, we need to take into account the expert's uncertainty.
- To computers, a natural language is the language of numbers, this is what they were deigned for.
- So, a natural idea is to describe the expert's degree of certainty in a statement by a number.

2. Need to describe experts' uncertainty in computer-based systems (cont-d)

- In a computer, 0 usually means "false" and 1 usually means "true".
- Thus, it makes sense to describe intermediate degrees of certainty by numbers between 0 and 1.

3. Need for aggregation operations

- When a computer-based system makes a recommendation or a conclusion, it usually uses several statements $S_1, \ldots, S_n$.
- Our degree of confidence in this recommendation is equal to our confidence that all these statements are true: $S_1 \,\&\, \ldots \,\&\, S_n$.
- In some cases, several different statements S_i lead to the same conclusion.
- In this case, we are interested in our degree of confidence that one of them is true, i.e., that $S_1 \vee \ldots \vee S_n$.
- There are exponentially many different combinations of statements: 2^N, when we have N statements.
- It is not feasible to ask the expert about all possible combinations.

4. Need for aggregation operations (cont-d)

- So, we need to estimate our degree of confidence in $A \,\&\, B$ and $A \vee B$ based on the degrees of confidence a and b in A and B.

- The corresponding algorithms $f_\&(a,b)$ and $f_\vee(a,b)$ are known as "and"- and "or"-operations.

5. Most systems use associative operations, but sometimes non-associative ones work better

- In general, the statement $(A \,\&\, B) \,\&\, C$ and $A \,\&\, (B \,\&\, C)$ mean the same.
- So, it is reasonable to select $f_{\&}(a,b)$ for which the estimates degree of confidence in these statement are equaL:

$$f_{\&}(f_{\&}(a,b),c) = f_{\&}(a, f_{\&}(b,c)).$$

- In mathematics, such operations are known as associative.
- There are many such associative operations: $\min(a,b)$, $a \cdot b$, $\max(a+b-1,0)$, etc.
- A recent study showed that in some risk-related applications, we get better results with non-associative operations

$$\frac{\max(a+b-1,0)+\min(a,b)}{2} \text{ and } \frac{a \cdot b + \min(a,b)}{2}.$$

- In this talk, we explain this empirical success.

6. Our explanation of the first successful formula

- The degrees of confidence can be viewed as subjective probability that a statement is true.
- If we know the probabilities a and b of two events, then the probability c that they both occur can take any value from the interval

$$[\underline{c}, \overline{c}] = [\max(a+b-1, 0), \min(a, b)].$$

- Thus, the expected utility $u = c \cdot u_0$ of making a decision based on $A \,\&\, B$ can take any value from

$$[\underline{u}, \overline{u}] = [\max(a+b-1, 0) \cdot u_0, \min(a, b) \cdot u_0].$$

- When we know the exact utilities of different decisions, we naturally select the one with the largest gain.
- In our case, we only know the interval of possible values of utility.
- In such cases, decision theory recommends to compare the combinations (first proposed by the Nobelist Hurwicz):

$$u = \alpha \cdot \overline{u} + (1-\alpha) \cdot \underline{u}.$$

7. Our explanation of the first successful formula (cont-d)

- In general, the recommended value is $\alpha = 0.5$.
- This leads to $u = c_0 \cdot u_0$, where

$$c_0 = \frac{\max(a + b - 1, 0) + \min(a, b)}{2}.$$

- This explains the first empirically successful formula.

8. Comment

- In general, this operation is non-associative.
- This means that there is a difference between estimates for $(A \,\&\, B) \,\&\, C$ and $A \,\&\, (B \,\&\, C)$.
- A paper by Trejo et al. showed that for this formula, the largest value of this difference is $1/9$.
- So, to make sure that this difference does not affect the decisions, it makes sense to consider $(1/9)$-close degree as similar.
- This way, we, in effect, only consider 9 possible degrees.
- This explains why humans are most comfortable with ≤ 9 items to choose from.
- This is an important part of the 7 ± 2 law well known in psychology.

9. Our explanation of the second successful formula

- Usually, there is a positive correlation between different experts.
- In situations when A and B are positively correlated, we have a narrower interval of c-values:

$$[\underline{c}, \overline{c}] = [a \cdot b, \min(a, b)].$$

- This leads to the narrower utility interval

$$[\underline{u}, \overline{u}] = [a \cdot b \cdot u_0, \min(a, b) \cdot u_0].$$

- For this interval, Hurwicz's approach leads to

$$u = \frac{\underline{u} + \overline{u}}{2} = c_0 \cdot u_0, \text{ where}$$

$$c_0 = \frac{a \cdot b + \min(a, b)}{2}.$$

- This explains the second empirically successful formula.

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