Positive vs. negative fallacy

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1. Formulation of the problem

- There are diseases for which no cure is known, some of them even fatal.
- Suppose now that a new medicine has been proposed that cures some of the patients with this disease.
- This medicine is not perfect – no medicine is.
- Shall we approve this medicine?
- Let us assume that in the tests, 30% of patients recovered.
- We can described it in two different ways:
  - With this medicine, 30 out of 100 patients will recover.
  - With this medicine, 70 out of 100 patients will remain sick.
- These two statements are logically equivalent.
- However, the first statement is perceived as more positive and preferable.

2. Formulation of the problem (cont-d)

- Experiments show that when presented with the first statement, people are more willing to vote for approval.
- How can we explain this difference?

3. Our main idea

- Our main idea is that our degree of confidence decreases with each logical inference step.
- So, while the statements are equivalent:
  - based on the second statement,
  - our degree of confidence in the conclusion – that the first statement is true – is smaller.
- This can also explain phenomena like heap paradox:
  - one grain of sand does not make a heap;
  - if you add one grain to something that it not heap it is still not a heap,
  - so strictly speaking by induction we should conclude that no heaps are possible,
  - but heaps *are* possible.

4. Let us consider an extreme example

- Let us consider an even more extreme example.
- Let us consider a diseases for which there is, at present, no cure.
- At present, all the patients who have this disease die.
- Suppose now that we have a new medicine that enables us to save the lives of 10 patients out of 100.
- If we package it this way, everybody will vote to approve this medicine.
- But from the logical viewpoint, it is possible to present the same statement in a completely different form.
- Namely, we say that 90 out of 100 patients who use this medicine will die.
- It's exactly the same statement, but somehow, when people hear it, they view it as negative.
- How can we explain this difference in rational terms?

5. Why is this a paradox

- In both cases:
  - we have a positive outcome, that 10 people will survive, and
  - and we gave a negative outcome, that 90 people die.
- So why is there the difference?
- This is surprising, because in mathematics:
  - no matter how many steps you make in following some logical reasoning,
  - the resulting statement is still correct.

6. But real life is not mathematics

- In real life, our reasoning is not perfect.
- When we say “if A, then B,” we don’t mean it as a mathematical truth.
- We mean that this is true with a high degree of confidence.
- This degree of confidence many be high, but it is never 100%.
- And important, that confidence decreases with each logical step.

7. This can explain the positive-negative paradox

- Let us show how this can explain the positive-negative paradox.
- Suppose that a person says that 10% of patients will survive.
- This is a direct statement, so we take it at face value.
- We can definitely deduce from it that 90% will die.
- However, this negative statement is indirect.
- It comes as a result of logical reasoning.
- So, we have smaller trust in the deduced statements than in the original statement.
- Thus, the effect of the positive statement is high, while the effect of the negative conclusion is lower.
- Suppose now that we say that 90% of the patients will die.
- Then the negative part is taken at face value, viewed with full confidence.

8. This can explain the positive-negative paradox (cont-d)

- However, the positive effect – that 10% will survive – requires an inferential step.
- So our confidence in the positive effect is smaller.
- In this case, the effect of the negative statement is high, while the effect of the positive statement is lower.

9. Let us describe this in mathematical terms

- The key concept that we will use is the concept of utility.
- In general, in English and in other languages, the word *utility* is vague.
- However, in decision theory, utility has a very precise definition.
- This notion is used to describe people's preferences.
- At first glance, this is difficult.
- How do you compare something like preferring ice cream over a burrito or vice versa?
- The way to do it is to establish a scale.

10. The formal notion of utility

- The scale used in decision theory has two reference points:
  - a very negative outcome A_-, such as having no money, and
  - a very positive outcome A_+, such as receiving \$100.
- Then, for any number p between 0 and 1, we can construct a lottery in which a person:
  - receives \$100 with probability p, and
  - receives nothing with the remaining probability $1 - p$.
- How do we find the utility of any other outcome – say, eating your favorite ice cream?
- For this purpose, we need to find the probability p for which this lottery has the same value as the ice cream.
- This probability p is your utility for this particular outcome.

11. Example

- To make this concrete, let's walk through an example.
- Suppose your favorite ice cream is chocolate chip cookie dough.
- We want to find the person's utility for this ice cream.
- We start by offering a choice: would you prefer the ice cream for certain, or \$100 with probability 0.5?
- At such high probability most people would take the \$100.
- So we know the utility is somewhere in the interval between 0 and 0.5.
- We then take a midpoint 0.25, and repeat the same question, but this probability 0.25.
- As we lower the probability, at some point, the lottery starts to feel less attractive than simply getting the ice cream.
- The probability at which you switch – the point where the two options feel exactly equivalent – is your utility for that ice cream.

12. Example (cont-d)

- When we tried it, that crossover happened somewhere between 12% and 13%.
- So the utility of that ice cream is approximately 0.125.

13. Utility of an action

- This is what utility means in decision theory:
  - not a vague sense of how much you like something,
  - but a precise probability
  - the one at which a lottery over the best and worst outcomes becomes exactly equivalent to the thing you are measuring.
- Now suppose you have some action whose outcome is uncertain:
  - with probability p_1 you get an outcome with utility u_1,
  - with probability p_2 you get an outcome with utility u_2,
  - and so on up to p_n and u_n.
- How can we find the overall utility of this action?
- Let us recall that each utility u_i represents the probability at which a lottery over A_+ and A_- is equivalent to the desired outcome.
- So, without losing equivalence, we can replace each outcome with its corresponding lottery.

14. Utility of an action (cont-d)

- When we do this, the whole action reduces to a single lottery over the best outcome A_+ and the worst outcome A_-.
- The overall probability of reaching A_+ is as follows:

$$p_1 \cdot u_1 + p_2 \cdot u_2 + \ldots + p_n \cdot u_n.$$

- This is the utility of the action.
- In mathematical terms, it is exactly the expected value of the utility.
- This gives us a precise, formal formula for measuring the utility of any action under uncertainty.

15. How does reasoning affect utility?

- So, how does reasoning affect utility?
- With any inferential step, utility decreases.
- We can represent this as a function $F(u)$ that maps the original utility u to a smaller after-reasoning value.
- The key requirement is that this function behaves consistently.
- Namely, decreasing each individual outcome's utility u_i should be equivalent to decreasing the utility u of the whole action:

$$F(p_1 \cdot u_1 + p_2 \cdot u_2 + \ldots + p_n \cdot u_n) = p_1 \cdot F(u_1) + p_2 \cdot F(u_2) + \ldots + p_n \cdot F(u_n).$$

- Let us show that this requirement leads to the linear formula $F(u) = c \cdot u$ for some constant c.

16. Derivation of a linear formula for the utility decrease

- Indeed, let us consider a simple lottery with two outcomes:
  - we receive the best outcome with probability p, and
  - we receive the worst outcome with probability $1 - p$.
- The utility of this lottery is $p \cdot 1 + (1 - p) \cdot 0 = p$.
- The result of applying the function F to this utility should be equivalent to applying F to each part individually:

$$F(p \cdot 1 + (1 - p) \cdot 0) = p \cdot F(1) + (1 - p) \cdot F(0).$$

- Since $F(0) = 0$, this simplifies to: $F(p) = p \cdot F(1)$.
- Since $F(1)$ is just a constant – we can denote it by c – we get: $F(u) = c \cdot u$.
- This confirms that each inferential step scales utility down by a fixed constant factor.

17. Derivation of a linear formula for the utility decrease (cont-d)

- The linear formula thus follows naturally and directly from the consistency requirement.
- And this is what ultimately explains the positive-negative paradox – and relates it to phenomena like the heap paradox

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