How to Make a Decision under Interval Uncertainty If We Do Not Know the Utility Function

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1. Formulation of the problem

- According to decision theory, decisions of a rational person are described by a function $u(a)$ called *utility*.
- An alternative in which we gain amount a is better than the alternative in which we gain amount b if and only if $u(a) > u(b)$.
- For the case of gain, the utility function is (non-strictly) increasing: if $a \le b$ then $u(a) \le u(b)$.
- In practice, we only know the consequence of each action with uncertainty.
- In many cases, all we know is the bounds on possible gain, i.e., the interval $[\underline{a}, \overline{a}]$ of possible values of the gain.

2. Formulation of the problem (cont-d)

- In this case, according to decision theory, the decision maker should:
  - select some value $\alpha \in [0, 1]$ describing the decision maker's degree of optimism-pessimism – and
  - select an alternative for which the value $\alpha \cdot u(\overline{a}) + (1 - \alpha) \cdot u(\underline{a})$ is the largest.
- Sometimes, we do not know the utility function.
- When can we still conclude that $[\underline{a}, \overline{a}]$ is better than $[\underline{b}, \overline{b}]$?
- The answer is easy:
  - when $\alpha = 1$ – then we select the alternative with the larger $\overline{a}$ – and
  - when $\alpha = 0$ – then we select the alternative with the larger $\underline{a}$.
- But what if $0 < \alpha < 1$?

3. Main result

For every $\alpha \in (0,1)$ and for every two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$, the following two conditions are equivalent:

1. $\alpha \cdot u(\overline{a}) + (1-\alpha) \cdot u(\underline{a}) \ge \alpha \cdot u(\overline{b}) + (1-\alpha) \cdot u(\underline{b})$ *for all non-strictly increasing functions* $u(a)$*;*

2. $\underline{a} \ge \underline{b}$ *and* $\overline{a} \ge \overline{b}$*.*

4. Proof

- Condition 2. implies condition 1. due to monotonicity.
- Let us prove that:
  - if the condition 2. is not satisfied, i.e., if $\underline{a} < \underline{b}$ or $\overline{a} < \overline{b}$,
  - then the condition 1. is violated for some increasing function $u(a)$.
- Indeed, if $\underline{a} < \underline{b}$, then we can take $u(a) = 0$ for $a \le \underline{a}$ and $u(a) = 1$ otherwise.
- Then, since $\overline{b} \ge \underline{b} > \underline{a}$, the right-hand side of the inequality 1. is equal to $\alpha + (1 - \alpha) = 1$.
- The left-hand side is equal to $\alpha \cdot u(\overline{a}) \le \alpha < 1$.
- So the inequality is not satisfied.

5. Proof (cont-d)

- If $\overline{a} < \overline{b}$, then we can take $u(a) = 0$ for $a < \overline{b}$ and $u(a) = 1$ otherwise.
- Then, since $\underline{a} \le \overline{a} < \overline{b}$, the left-hand side of 1. is equal to 0.
- The right-hand side is larger than or equal to $(1-\alpha)\cdot u(\overline{b}) = 1-\alpha > 0$.
- So the inequality is not satisfied either. The result is proven.

6. Comment

- We considered the case when we know α and but we do not know $u(a)$.
- Similar results can be proven in two other cases:
  - when we know $u(a)$ – which is strictly increasing – but we do not know α;
  - when we do not know neither α nor $u(a)$.
- For example, when we do not know α, then we can have $\alpha = 0$ and $\alpha = 1$.
- In this case we also have $\underline{a} \geq \underline{b}$ and $\overline{a} \geq \overline{b}$.

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