In Practice, Estimates Based on Gaussian Uncertainty Are More Accurate Than Interval Estimates

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1. How can we describe measurement uncertainty: probabilistic approach

- In many practical situations, the measurement error is caused by many independent small factors.
- It is known that in this case, the resulting probability distribution is close to Gaussian (normal).
- This is known as the Central Limit Theorem.
- It is therefore reasonable to describe measurement errors as normally distributed random variables.
- A normal distribution is uniquely determined by its mean m and its standard deviation σ.
- Both can be estimated based on a few test measurements.

2. How can we describe measurement uncertainty: probabilistic approach (cont-d)

- Once we know the mean (known as *bias*), we can:
  - subtract it from all measurement results, and
  - conclude that the mean value of the resulting measurement error is 0.
- To increase accuracy, a natural idea is to perform several (n) measurements and take the arithmetic average.
- Then the standard deviation of the resulting estimate is $\sigma/\sqrt{n}$.

3. Interval uncertainty

- Strictly speaking, for a normal distribution, any value is possible – just probabilities of large values are very small.
- In practice, we ignore these small probabilities and assume:
  - that the absolute value of the measurement error is always smaller than $\Delta \stackrel{\text{def}}{=} k \cdot \sigma$, where $k = 2$, 3, or 6,
  - i.e., in effect, that the probability distribution is limited to the interval $[-\Delta, \Delta]$.
- In this case:
  - after the measurement results in a value $\widetilde{x}$,
  - we conclude that the actual (unknown) value x is in the interval

$$[\widetilde{x} - \Delta, \widetilde{x} + \Delta].$$

4. Interval uncertainty (cont-d)

- When we measure several times, we conclude that x is in the intersection of the corresponding intervals.
- Interestingly, the large n, the width of this intersection interval decreases as $1/n$.
- It decreases much faster than the k-σ interval corresponding to $\sigma/\sqrt{n}$ whose width decreases much slower – as $1/\sqrt{n}$.

5. A natural question and what we do in this talk

- A natural question that we answer in this talk is: Which estimates are better in practice, for realistic values n?

6. Analysis of the problem and the resulting answer

- It is known that:
  - if we want the bound of the intersection interval with confidence p_0 for some $p_0 \approx 1$,
  - we get the bound which is asymptotically equal to $\dfrac{A}{n}$,
  - where $A = -\dfrac{2}{\rho} \cdot \ln\left(\dfrac{1-p_0}{2}\right)$ and ρ is the probability density at the point Δ.
- For statistical estimate, the bound of the resulting k-σ interval is equal to $\dfrac{\Delta}{\sqrt{n}}$.
- These values become equal when $\dfrac{A}{n} = \dfrac{\Delta}{\sqrt{n}}$, i.e., when $n = \left(\dfrac{A}{\Delta}\right)^2$.
- For smaller n, the Gaussian interval is narrower.

7. Analysis of the problem and the resulting answer (cont-d)

- For the k-σ interval, we have $\rho = \dfrac{1}{\sqrt{2\pi}\cdot\sigma}\cdot\exp\left(-\dfrac{k^2}{2}\right)$, so

$$\frac{A}{\Delta} = -2\cdot\sqrt{2\pi}\cdot k\cdot\exp(k^2/2)\cdot\ln((1-p_0)/2)).$$

- In particular, for $k=2$, when $p_0=0.95$, we get $n=(A/\Delta)^2\approx 75\,000$.

8. Conclusion

- In all realistic cases, we have $n \ll 75\,000$.
- So, the Gaussian estimate is still better.

9. Reference

- G. W. Walster and V. Kreinovich, "For unknown–but–bounded errors, interval estimates are often better than averaging", *ACM SIGNUM Newsletter*, 1996, Vol. 31, No. 2, pp. 6–19.

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