In Practice, Estimates Based on Gaussian Uncertainty Are More Accurate Than Interval Estimates

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1. How can we describe measurement uncertainty: probabilistic approach

- In many practical situations, the measurement error is caused by many independent small factors.
- It is known that in this case, the resulting probability distribution is close to Gaussian (normal).
- This is known as the Central Limit Theorem.
- It is therefore reasonable to describe measurement errors as normally distributed random variables.
- A normal distribution is uniquely determined by its mean $m$ and its standard deviation $\sigma$.
- Both can be estimated based on a few test measurements.
2. How can we describe measurement uncertainty: probabilistic approach (cont-d)

- Once we know the mean (known as bias), we can:
  - subtract it from all measurement results, and
  - conclude that the mean value of the resulting measurement error is 0.

- To increase accuracy, a natural idea is to perform several \((n)\) measurements and take the arithmetic average.

- Then the standard deviation of the resulting estimate is \(\sigma/\sqrt{n}\).
3. Interval uncertainty

- Strictly speaking, for a normal distribution, any value is possible – just probabilities of large values are very small.
- In practice, we ignore these small probabilities and assume:
  - that the absolute value of the measurement error is always smaller than $\Delta \stackrel{\text{def}}{=} k \cdot \sigma$, where $k = 2$, 3, or 6,
  - i.e., in effect, that the probability distribution is limited to the interval $[-\Delta, \Delta]$.
- In this case:
  - after the measurement results in a value $\tilde{x}$,
  - we conclude that the actual (unknown) value $x$ is in the interval $[\tilde{x} - \Delta, \tilde{x} + \Delta]$. 
4. Interval uncertainty (cont-d)

- When we measure several times, we conclude that $x$ is in the intersection of the corresponding intervals.

- Interestingly, the large $n$, the width of this intersection interval decreases as $1/n$.

- It decreases much faster than the $k$-$\sigma$ interval corresponding to $\sigma/\sqrt{n}$ whose width decreases much slower – as $1/\sqrt{n}$. 

5. A natural question and what we do in this talk

- A natural question that we answer in this talk is: Which estimates are better in practice, for realistic values $n$?
6. Analysis of the problem and the resulting answer

- It is known that:
  - if we want the bound of the intersection interval with confidence $p_0$ for some $p_0 \approx 1$,
  - we get the bound which is asymptotically equal to $\frac{A}{n}$,
  - where $A = -\frac{2}{\rho} \cdot \ln \left( \frac{1 - p_0}{2} \right)$ and $\rho$ is the probability density at the point $\Delta$.

- For statistical estimate, the bound of the resulting $k$-$\sigma$ interval is equal to $\frac{\Delta}{\sqrt{n}}$.

- These values become equal when $\frac{A}{n} = \frac{\Delta}{\sqrt{n}}$, i.e., when $n = \left( \frac{\Delta}{A} \right)^2$.

- For smaller $n$, the Gaussian interval is narrower.
7. Analysis of the problem and the resulting answer (cont-d)

- For the $k$-$\sigma$ interval, we have $\rho = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot \exp \left( -\frac{k^2}{2} \right)$, so

$$\frac{A}{\Delta} = -2 \cdot \sqrt{2\pi} \cdot k \cdot \exp(k^2/2) \cdot \ln((1 - p_0)/2)).$$

- In particular, for $k = 2$, when $p_0 = 0.95$, we get $n = (A/\Delta)^2 \approx 75000$. 
8. Conclusion

- In all realistic cases, we have $n \ll 75\,000$.
- So, the Gaussian estimate is still better.
9. Reference

10. Acknowledgments

This work was supported in part by:

- National Science Foundation grants 1623190, HRD-1834620, HRD-2034030, and EAR-2225395;
- AT&T Fellowship in Information Technology;
- program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and
- a grant from the Hungarian National Research, Development and Innovation Office (NRDI).