Finite Fields – A Possible Way to Avoid Infinities in Physical Computations

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1. Infinities in physics: a problem

- Modern physics has many successes.
- However, there are still many cases when known equations leads to meaningless infinite values for physical quantities.
- Example: what is the overall energy of an electron – including:
  - the energy $E = m_0 \cdot c^2$ corresponding to its mass m_0 and
  - the energy the electron's electromagnetic field?
- Straightforward calculations lead to infinity.
- There are tricks – called *renormaliztaion* – that enables physicists to avoid infinities.
- E.g., we can assume that m_0 is close to $-\infty$ and tend to a limit.
- However, it is desirable to avoid infinities without adding special tricks.

2. Finite fields: a possible approach

- Many physical quantities are discrete.
- For example, electric charge can only be proportional to the electron's charge – i.e., is described by an integer.
- For charges, addition makes physical systems: when we bring two objects together, their charges add.
- However, it does not necessarily mean that we need to consider infinities.
- For example, for an electric meter, once the number reaches a certain threshold, it turns back to 0.

3. Finite fields: a possible approach (cont-d)

- In general, for any prime number p, remainders modulo p with the usual addition-modulo-p and multiplication-modulo-p operations:
  - form what in mathematics is called a finite field,
  - with usual relation between addition and multiplication and
  - with the possibility of dividing by any non-zero number.
- The set of all such remainders is usually denoted by

$$Z/pZ = \{0, 1, \ldots, p-1\}.$$

4. What we do in this talk

- In this talk, we analyze how this idea affects the usual division of numbers into small (S), medium (M), and large (L).
- For example, we can consider all values < 0.1 as small, all values > 10 as large, and all others as medium.
- In general, commonsense implies that if x is small, then $1/x$ is large, and vice versa.
- For a usual real line:
  - no matter what thresholds we choose,
  - some numbers are so large that they cannot be represented as a product of two small or medium numbers.
- In the above example, such is any number larger than 100.
- Interestingly, in the finite field case, this conclusion is no longer valid.

5. Main result

- Suppose that Z/pZ is divided into three disjoint sets S, M, and L, for which, for every x, $x \in S$ if and only if $1/x \in L$.
- Then, every element $x \in L$ can be represented as a product $x = a \cdot b$ of two numbers $a, b \in S \cup M$.

6. Proof

- Number 1 cannot be small, since then we would have $1/1 = 1 \in L$ but $S \cap L = \emptyset$.
- Similarly, 1 cannot be large, so $1 \in M$.
- So, $\leq p - 1$ elements are small or large.
- Small and large numbers are in 1-1 correspondence via $x \mapsto 1/x$, and a number cannot be both small and large.
- So the number of large numbers is $\leq (p-1)/2$.
- Thus, the number of small or medium numbers is at least

$$p - (p-1)/2 = (p+1)/2.$$

- Let us take any large number ℓ and let us consider ratios ℓ/x for all $x \in S \cup M$.
- There are $\geq (p+1)/2$ numbers in $S \cup M$, so we will have $\geq (p+1)/2$ different ratios.

7. Proof (cont-d)

- These ratios cannot be all large, since there are $\leq (p-1)/2$ large numbers.
- Thus, at least of these ratios ℓ/x_0 is in $S \cup M$.
- For this ratio, we have the desired representation $\ell = x_0 \cdot (\ell/x_0)$.

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