Paradox of Causality and Paradoxes of Set Theory

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1. Paradox of causality: reminder

- It is well known that time travel can lead to paradoxes:
  - If a person A travels to the past and kills his own grandfather before his own father was conceive,
  - there is no possibility for A to be born –
  - but he was actually born.
- In general, this paradox appears every time we close a closed loop of causality relations:
  - whether we have event e_1 causally affecting event e_2 (we will denote it by $e_1 < e_2$) and e_2 causally affecting event e_1 ($e_2 < e_1$),
  - or whether we have $e_1 < e_2$, $e_2 < e_3$, and $e_3 < e_1$,
  - or whether we have a even longer loop.

2. Paradox of intuitive ("naive") set theory

- In set theory, a known paradox – first discovered by Bertran Russell – is related:
  - to the possibility of having a simple element-of loop,
  - i.e., the possibility to have $x \in x$ for some set x.
- Specifically, the paradox appears when we consider the set $S = \{x : x \notin x\}$ of all the sets that are not elements of themselves.
- The paradox appears when we check whether $S \in S$.
- Indeed, we have either $S \in S$ or $S \notin S$, and in both cases, we get a contradiction.
- if $S \in S$, then, by definition of the set S, its element S must have the property that defines this set, i.e., we must have $S \notin S$.
- This contradicts to our assumption that $S \in S$;

3. Paradox of intuitive ("naive") set theory (cont-d)

- On the other hand, if $S \notin S$, then, by definition of the set S, the set S does not have the property that defines this set, i.e., we have $S \in S$.
- This contradicts to our assumption that $S \notin S$.

4. Natural idea

- Both paradoxes relate to close loops; the main difference is that:
  - the causality paradox appears no matter how long is the loop, while
  - the corresponding paradox of set theory is only known to appear when we consider a on-element loop: $x \in x$.
- It is therefore reasonable to check whether a similar paradox appears in set theory when we consider loops of arbitrary length.

5. Main result

- In this talk, we show that such an extension is indeed possible.
- In other words, it is possible to formulate a similar paradox related to a loop $x \in x_1 \in x_2 \in \ldots \in x_n \in x$, for any n.
- Indeed, let us consider the set

$$S_n = \{x : \neg\exists x_1, \ldots, x_n (x \in x_1 \in x_2 \in \ldots \in x_n \in x)\}.$$

- Then, we have either $S_n \in S_n$ or $S_n \notin S_n$ – and in both cases, we get a contradiction.
- If $S_n \in S_n$, then means that we cannot have sets $x_1, \ldots, x_n$ for which $S_n \in x_1 \in \ldots \in x_n \in S_n$.
- However, we *do* have such sets if we take $x_2 = \ldots = x_n = S_n$ – a contradiction.

6. Main result (cont-d)

- On the other hand, if $S_n \notin S_n$, then there should exist a sequence x_1, …, x_n for which $S_n \in x_1 \in \ldots \in x_n \in S_n$.
- In particular, this means that $x_n \in S_n$.
- So, for the element x_n, we have a loop $x_n \in S_n \in x_1 \ldots \in x_{n-1} \in x_n$.
- This means, by the definition of the set S_n, that x_n cannot be the element of S_n – also a contradiction.

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