How to compare situations in which we measure different quantities with different uncertainty

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1. Formulation of the problem

- In many practical situations, we need to design a device for measuring several quantities.
- For example, we want to send it on a space mission.
- Often, this is largely a mission to the unknown – we do not know a priori which measurements will be more important.
- In the ideal world, we should measure each of the quantities of interest with maximum accuracy.
- However, in practice, there are limits on the size and weight of the device.
- So our options are limited.
- Under such restrictions, we may have different possible sets of accuracies $a = (a_1, \ldots, a_n)$ for measuring the desired n quantities.
- Which options should we select?

2. Let us formulate this problem in precise terms

- To select the best option, we need to describe the relation "better of or the same quality".
- We will denote by $a \succeq b$.
- This relation should be reflexive ($a \succeq a$) and transitive (if $a \succeq b$ and $b \succeq c$, then $a \succeq c$).
- Of course, if all measurement are more accurate, i.e., if $a_i \le b_i$ for all i, then we should have $a \succeq b$.
- And if also $a_i < b_i$ for some i, we should have $b \not\succeq a$.
- Since we do not know which quantity is more important, the relation should not change if we swap some quantities.
- In precise terms, for each permutation π, if $a \succeq b$, then we should have $\pi(a) \succeq \pi(b)$.

3. Let us formulate this problem in precise terms (cont-d)

- Finally, the selection should not depend on what measuring unit we use for each quantity.
- For example, for measuring length, we can use meters or centimeters.
- If we switch to a unit which is λ_i times smaller, all numerical values are multiplied by λ_i.
- Thus, for each tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$, if $(a_1, \ldots, a_n) \succeq (b_1, \ldots, b_n)$, then we should have $(\lambda_1 \cdot a_1, \ldots, \lambda_n \cdot a_n) \succeq (\lambda_1 \cdot b_1, \ldots, \lambda_n \cdot b_n)$.
- This is known as *scale-invariance.*

4. Our result: formulation and meaning

- It turns out that for permutation-invariant and scale-invariant relations, $a \succeq b$ is equivalent to $a_1 \cdot \ldots \cdot a_n \le b_1 \cdot \ldots \cdot b_n$.
- This result has the following natural interpretation.
- If we start with some area of values of size $X = X_1 \times \ldots \times X_n$, then:
  - we have X_1/a_1 possible different measured values of x_1, etc.,
  - with the total of $N = (X_1/a_1) \cdot \ldots \cdot (X_n/a_n)$ combinations.
- Here, $N = X/(a_1 \cdot \ldots \cdot a_n)$.
- So, the smaller the product of a_i's, the more alternatives we get and thus, the more information we gain about the studied object.

5. Proof

- For $n = 2$, for all a_1 and a_2, due to permutation-invariance, we have $(\sqrt{a_1}, \sqrt{a_2}) \sim (\sqrt{a_2}, \sqrt{a_1})$.
- Here $a \sim b$ means $a \succeq b$ and $b \succeq a$.
- For $\lambda_1 = \sqrt{a_1}$ and $\lambda_2 = \sqrt{a_2}$, scale-invariance implies that $(a_1, a_2) \sim (\sqrt{a_1 \cdot a_2}, \sqrt{a_1 \cdot a_2})$.
- So, by transitivity, if the two options have the same product $a_1 \cdot a_2$, they are equivalent.
- For $n > 2$, we can similarly prove that:
  - if we replace two values a_i and a_{i+1} with another two values with the same product,
  - the options remain equivalent.
- Thus, if we start with any option $a = (a_1, \ldots, a_n)$ with $s \stackrel{\text{def}}{=} \sqrt[n]{a_1 \cdot \ldots \cdot a_n}$, then we first replace a_1 and a_2 with s and $(a_1 \cdot s_2)/s$.

6. Proof (cont-d)

- Then, for each k:
  - once we have an equivalent option $(s, \ldots, s, a'_{k+1}, a'_{k+2}, \ldots, a'_n)$,
  - we replace a'_{k+1} and a'_{k+2} with s and $(a'_{k+1} \cdot a'_{k+2})/s$, etc.
- At the end, we will be able to conclude that the original option a is equivalent to $(s, \ldots, s)$.
- For such options, the smaller s, the better – and if s is smaller, then s^n is also smaller.
- Thus, the relative quality of different options is indeed determined by the product s^n of their accuracies:
  - the smaller this product,
  - the better.

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