Orevkov, Khalfin, and Quantum Field Theory: How Constructive Mathematics Can Help Physics

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1. Orevkov's 1972 Results

- In 1972, Vladimir Orevkov presented a talk on constructive complex analysis at LOMI.
- The main results from this talk were published in 1974.
- In that paper, he provided new more explicit constructive proofs of basic results of complex analysis:
  - that a function is differentiable iff it can be expanded in Taylor series at each point,
  - that two such (analytical) functions are equal if they coincide on a non-finite compact set, and
  - that it is possible to constrictively find all the roots of such function on each bounded domain.
- These results were previously proved by Vladimir Lifschitz in a more implicit way.

2. Orevkov's 1972 Results

- As usual, many results from classical (non-constructive) mathematics turned out to be constructively true.
- Some results from classical mathematics turned out to be constructively false, in the sense that:
  - while there is a classical existence theorem,
  - no general algorithm for constructing the corresponding object is possible.

3. Can This Result Help Physics?

- This talk attracted attention of Leonid Khalfin, Orevkov's LOMI colleague interested in physics applications.
- Khalfin asked whether constructive mathematics can solve a problem related to physics use of complex #s.
- On macro-level, we observe many non-smooth and even discontinuous phenomena:
  - earthquakes,
  - phase transitions, etc.
- However, on the micro-level, all equations and all phenomena are smooth – and even analytical.
- Some of these phenomena are very fast – so we perceive them as discontinuous.

4. Can This Result Help Physics (cont-d)

- For complex numbers, smoothness means analyticity.
- Analyticity has been successfully used in quantum field theory.
- For example, to compute the values of some integral expressions, it is convenient to use the fact that:
  - for an analytical function,
  - a contour integral over a closed loop is 0:

$$\int_\gamma f(z)\,dz = 0$$

  - or it is equal to an explicit expression in terms of the poles.

5. Can This Result Help Physics (cont-d)

- Thus, by using a loop $[-N, N] \cup \gamma'$, we can:
  - replace a difficult-to-compute integral over real numbers $\int_{-N}^{N} f(x)\,dx$
  - with an easier-to-compute integral over the complex values $\int_{\gamma'} f(z)\,dz$.
- This idea – mostly pioneered by Nikolai Bogolyubov – led to many successful applications.
- This "macro" analyticity has been confirmed by many experiments and makes perfect physical sense.

6. Can This Result Help Physics (cont-d)

- The problem is that in traditional mathematics:
  - such "macro" analyticity is equivalent to "micro" one,
  - that the corresponding dependencies can be expanded in Taylor series:

$$f(z) = a_0 + a_1 \cdot (z - z_0) + a_2 \cdot (z - z_0)^2 + \ldots + a_n \cdot (z - z_0)^n + \ldots$$

- In the opinion of physicists, however:
  - this "micro" analyticity does not make direct physical sense,
  - since on the micro level, quantum uncertainty makes exact measurements impossible.

7. Can This Result Help Physics (cont-d)

- From this viewpoint, it is desirable to come up with a model in which:
  - physically meaningful macro analyticity is present, but
  - physically meaningless micro analyticity is not.
- Khalfin hoped that:
  - this "thornless rose" effect can be achieved
  - if we consider constructive mathematics instead of the traditional one.

8. Can This Result Help Physics (cont-d)

- In the early 1970s, this hope did not materialize, since:
  - as Errett Bishop has shown in his 1967 book (and as Vladimir Lifschitz pointed to Khalfin),
  - the fact that macro analyticity implies micro one can be proven in constructive mathematics as well.
- Indeed, once we know $f(z)$, we can determine all the coefficients a_n as

$$a_n = \frac{1}{2\pi \cdot \mathrm{i}} \cdot \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

- And there are known algorithms for computing an integral of a computable function.

9. Problem Revisited

- Bishop's derivation is based on the usual constructive mathematics.
- In this approach, existence of an object means, in effect:
  - the existence of an algorithm producing more and more accurate approximations to this object,
  - irrespective to how long this algorithm may take.
- A more realistic idea is to only allow feasible (= polynomial-time) algorithms are allowed.
- It turns out that in this case, Khalfin's dream *can* be materialized.

10. Problem Revisited (cont-d)

- Indeed: while there exists an algorithm computing:
  - for each computable macro analytical function,
  - all the terms in its Taylor series expansion.
- However, the computation time of this algorithm seems to grow exponentially with the number n of the term.
- Let us provide arguments for this conclusion.

11. Explanation

- We have a computable function $f(z)$.
- This means that we can, given z, compute $f(z)$.
- For simplicity, we can also assume that we know the upper bound D on $|f'(z)| \le D$.
- Computation of the n-th Taylor coefficient a_n is based on the formula
$$a_n = \frac{1}{2\pi \cdot \mathrm{i}} \cdot \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$
- Here, the simplest possible loop γ around the point z_0 is a circle of some small radius $r < 1$.
- For this loop, $|z - z_0| = r$.
- We want to compute a_n with a given accuracy $\varepsilon > 0$.
- This means that we need to compute the corresponding integral with accuracy $\varepsilon' = 2\pi \cdot \varepsilon$.

12. Explanation (cont-d)

- A natural way to compute an integral $\int g(z)\,dz$ is to consider the corresponding integral sum

$$\sum g(z_i)\cdot\Delta z, \text{ with } |z_{i+1}-z_i| = h \text{ for some small } h.$$

- In this approximation, we approximate $g(z)$ with $g(z_i)$ on each arc of length h for which $|z-z_i|\le h/2$.

- The inaccuracy of this approximation is

$$|g(z)-g(z_i)| \le \left(\max_z |g'(z)|\cdot|z-z_i|\right) \le$$

$$\max_z |g'(z)|\cdot(h/2).$$

- Here, $g(z) = \dfrac{f(z)}{(z-z_0)^{n+1}} \approx \dfrac{f(z)}{r^{n+1}}$.

- Thus, $\max_z |g'(z)| \le \dfrac{\max|f'(z)|}{r^{n+1}} = \dfrac{D}{r^{n+1}}$.

13. Explanation (cont-d)

- So, the approximation accuracy is $\dfrac{D}{r^{n+1}} \cdot (h/2)$.
- To get accuracy ε', we need to take h for which

$$\frac{D}{r^{n+1}} \cdot (h/2) = \varepsilon', \text{ i.e., } h = 2\frac{\varepsilon'}{D} \cdot r^{n+1}.$$

- The whole loop γ of length $2\pi \cdot r$ should be covered by intervals of length h.
- These intervals correspond to values z_i at which we compute $f(z)$.
- Thus, we need to compute $f(z)$ for $N = \dfrac{2\pi \cdot r}{h}$ points.
- Substituting the above expression for h, we conclude that we need to compute $f(z)$ at

$$N = \frac{2\pi \cdot r \cdot D}{2\varepsilon' \cdot r^{n+1}} \sim r^{-n} \text{ points.}$$

14. Explanation (cont-d)

- We have shown that we need to compute $f(z)$ at

$$N = \frac{2\pi \cdot r \cdot D}{2\varepsilon' \cdot r^{n+1}} \sim r^{-n} \text{ points.}$$

- Since $r < 1$, this number indeed grows exponentially with n.

15. Possible Applications

- This result will probably be of interest to theoreticians like Khalfin interested:
  - in providing physical theories
  - with physically meaningful mathematical foundations.
- This result may also have practical applications if we take into account that:
  - many times when we encountered a physical process whose properties are difficult to compute,
  - it became possible to use this process to speed up computations.
- Successes of quantum computing are the latest example of this phenomenon.

16. Possible Applications (cont-d)

- From this viewpoint:
  - maybe measurement of the corresponding Taylor coefficients
  - can lead to yet another efficient quantum computing scheme?

17. Bibliography

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