Beyond Traditional Applications of Fuzzy Techniques: Main Idea and Case Studies

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1. Traditional Use of Fuzzy Logic

- Expert knowledge is often formulated by using imprecise ("fuzzy") from natural language (like "small").
- Fuzzy logic techniques was originally invented to translate such knowledge into precise terms.
- Such a translation is still the main use of fuzzy techniques.
- *Example:* we want to control a complex plant for which:
  - no good control technique is known, but
  - there are experts how can control this plant reasonably well.
- So, we elicit rules from the experts.
- Then we use fuzzy techniques to translate these rules into a control strategy.

2. Fuzzy Logic Can Help in Other Cases As Well

- Lately, it turned out that fuzzy techniques can help in another class of applied problems: in situations when
  - there are semi-heuristic techniques for solving the corresponding problems, i.e.,
  - techniques for which there is no convincing theoretical justification.
- These techniques lack theoretical justification.
- Their previous empirical success does not guarantee that these techniques will work well on new problems.
- Thus, users are reluctant to use these techniques.

3. Additional Problem of Semi-Heuristic Techniques

- Semi-heuristic techniques are often not perfect.
- Without an underlying theory, it is not clear how to improve their performance.
- For example, linear models can be viewed as first approximation to Taylor series.
- So, a natural next approximation is to use quadratic models.
- However, e.g., for ℓ^p-models:
  - when they do not work well,
  - it is not immediately clear what is a reasonable next approximation.

4. What We Show

- We show that in many such situations, the desired theoretical justification can be obtained if:
  - in addition to known (crisp) requirements on the desired solution,
  - we also take into account requirements formulated by experts in natural-language terms.
- Naturally, we use fuzzy techniques to translate these imprecise requirements into precise terms.
- To make the resulting justification convincing, we need to make sure that this justification works:
  - not only for one specific choice of fuzzy techniques (membership function, t-norm, etc.),
  - but for all techniques which are consistent with the practical problem.

5. Case Studies

As examples, we provide the detailed justification:

- of sparsity techniques in data and image processing –
  - a very successful hot-topic technique
  - whose success is often largely a mystery; and
- of ℓ^p-regularization techniques in solving inverse problems
  - an empirically successful alternative to smooth approaches
  - which is appropriate for situations when the desired signal or image is not smooth.

Part I

Why Sparse? Fuzzy Techniques Explain Empirical Efficiency of Sparsity-Based Data- and Image-Processing Algorithms

6. Sparsity Is Useful, But Why?

- In many practical applications, it turned out to be efficient to assume that the signal or an image is *sparse*:
  - when we decompose the original signal $x(t)$ (or image) into appropriate basic functions $e_i(t)$:
$$x(t) = \sum_{i=1}^{\infty} a_i \cdot e_i(t),$$
  - then most of the coefficients a_i in this decomposition will be zeros.
- It is often beneficial to select, among all the signals consistent with the observations, the signal for which
$$\#\{i : a_i \neq 0\} \to \min \text{ or } \sum_{i:a_i\neq 0} w_i \to \min .$$
- At present, the empirical efficiency of sparsity-based techniques remains somewhat a mystery.

7. Before We Perform Data Processing, We First Need to Know Which Inputs Are Relevant

- In general, in data processing, we:
  - estimate the value of the desired quantity y_j based on
  - the values of the known quantities $x_1, \ldots, x_n$ that describe the current state of the world.
- In principle, all possible quantities $x_1, \ldots, x_n$ could be important for predicting some future quantities.
- However, for each specific quantity y_j, usually, only a few of the quantities x_i are actually useful.
- So, we first need to check which inputs are actually useful.
- This checking is an important stage of data processing: else we waste time processing unnecessary quantities.

8. Analysis of the Problem

- We are interested in a reconstructing a signal or image $x(t) = \sum_{i=1}^{\infty} a_i \cdot e_i(t)$ based on:
  - the measurement results and
  - prior knowledge.
- First, we find out which quantities a_i are relevant.
- The quantity a_i is irrelevant if it does not affect the resulting signal, i.e., if $a_i = 0$.
- So, first, we decide which values a_i are zeros and which are non-zeros.
- Out of all such possible decisions, we need to select *the most reasonable one.*
- *Problem:* "reasonable" is not a precise term.

9. Let Us Use Fuzzy Logic

- *Reminder:* we want the most reasonable decision, but "reasonable" is not a precise term.
- So, to be able to solve the problem, we need to translate this imprecise description into precise terms.
- Let's use fuzzy techniques which were specifically designed for such translations.
- In fuzzy logic, we assign, to each statement S, our degree of confidence d in S.
- E.g., we ask experts to mark, on a scale from 0 to 10, how confident they are in S.
- If an expert marks the number 7, we take $d = 7/10$.
- Thus, for each i, we can learn to what extent $a_i = 0$ or $a_i \neq 0$ are reasonable.

10. Need for an "And"-Operation

- We want to estimate, for each tuple of signs, to which extent this tuple is reasonable.
- There are 2^n such tuples, so for large n, it is not feasible to ask about all of them.
- We thus need to estimate:
  - the degree to which a_1 is reasonable *and* a_2 is reasonable ...
  - based on individual degrees to which a_i are reasonable.
- In other words:
  - we know the degrees of belief $a = d(A)$ and $b = d(B)$ in statements A and B, and
  - we need to estimate the degree of belief in the composite statement $A \,\&\, B$, as $f_\&(a, b)$.

11. The "And"-Estimate Is Not Always Exact: an Example

- First case:
  - A is "coin falls heads", B is "coin falls tails", then for a fair coin, degrees a and b are equal: $a = b$.
  - Here, $A \,\&\, B$ is impossible, so our degree of belief in $A \,\&\, B$ is zero: $d(A \,\&\, B) = 0$.
- Second case:
  - If we take $A' = B' = A$, then $A' \,\&\, B'$ is simply equivalent to A.
  - So we still have $a' = b' = a$ but this time $d(A' \,\&\, B') = a > 0$.
- In these two cases:
  - we have $d(A') = d(A) = a$ and $d(B') = d(B) = b$,
  - but $d(A \,\&\, B) \neq d(A' \,\&\, B')$.

12. Which "And"-Operation (t-Norm) Should We Choose

- The corresponding function $f_\&(a, b)$ must satisfy some reasonable properties: e.g.,
  - since $A \,\&\, B$ means the same as $B \,\&\, A$, this operation must be commutative;
  - since $(A \,\&\, B) \,\&\, C$ is equivalent to $A \,\&\, (B \,\&\, C)$, this operation must be associative, etc.
- *Known result:* each such operation can be approximated, with any given accuracy,
  - by an *Archimedean* t-norm

$$f_\&(a, b) = f^{-1}(f(a) \cdot f(b)),$$

  - for some strictly increasing function $f(x)$.
- Thus, without losing generality, we can assume that the actual t-norm is Archimedean.

13. Let Us Use Fuzzy Logic

- Let $d_i^= \stackrel{\text{def}}{=} d(a_i = 0)$ and $d_i^{\neq} \stackrel{\text{def}}{=} d(a_i \neq 0)$.
- So, for each sequence $(\varepsilon_1, \varepsilon_2, \ldots)$, where ε_i is $=$ or $\neq$:

$$d(\varepsilon) = f_\&(d_1^{\varepsilon_1}, d_2^{\varepsilon_2}, \ldots).$$

- *Problem:*
  - out of all sequences ε which are consistent with the measurements and with the prior knowledge,
  - we must select the one for which this degree of belief is the largest possible.
- If we have no information about the signal, then the most reasonable choice is $x(t) = 0$, i.e.,

$$a_1 = a_2 = \ldots = 0 \text{ and } \varepsilon = (=, =, \ldots).$$

- Similarly, the least reasonable is the sequence in which we take all the values into account, i.e., $\varepsilon = (\neq, \ldots, \neq)$.

14. Definitions

- By a *t-norm*, we mean $f_\&(a,b) = f^{-1}(f(a)\cdot f(b))$, where $f : [0,1] \to [0,1]$ is continuous, $\uparrow$, $f(0) = 0$, $f(1) = 1$.
- By a *sequence*, we mean a sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$, where each symbol ε_i is equal either to $=$ or to $\neq$.
- Let $d^= = (d_1^=, \ldots, d_N^=)$ and $d^{\neq} = (d_1^{\neq}, \ldots, d_N^{\neq})$ be sequences of real numbers from the interval $[0,1]$.
- For each sequence ε, we define its *degree of reasonableness* as $d(\varepsilon) \stackrel{\text{def}}{=} f_\&(d_1^{\varepsilon_1}, \ldots, d_N^{\varepsilon_N})$.
- We say that the sequences $d^=$ and $d^{\neq}$ *properly describe reasonableness* if the following two conditions hold:
  - for $\varepsilon_= \stackrel{\text{def}}{=} (=, \ldots, =)$, $d(\varepsilon_=) > d(\varepsilon)$ for all $\varepsilon \neq \varepsilon_=$,
  - for $\varepsilon_{\neq} \stackrel{\text{def}}{=} (\neq, \ldots, \neq)$, $d(\varepsilon_{\neq}) < d(\varepsilon)$ for all $\varepsilon \neq \varepsilon_{\neq}$.
- For each set S of sequences, we say that a sequence $\varepsilon \in S$ *is the most reasonable* if $d(\varepsilon) = \max\limits_{\varepsilon' \in S} d(\varepsilon')$.

15. Main Result

- **Proposition.**
  - *Let us assume that the sequences* $d^=$ *and* $d^{\neq}$ *properly describe reasonableness.*
  - *Then, there exist weights* $w_i > 0$ *for which, for each set* S*, the following two conditions are equivalent:*
    - *the sequence* $\varepsilon \in S$ *is the most reasonable,*
    - *the sum* $\sum\limits_{i:\varepsilon_i = \neq} w_i = \sum\limits_{i:a_i \neq 0} w_i$ *is the smallest possible.*
- **Discussion:** thus, fuzzy-based techniques indeed naturally lead to the sparsity condition.

16. Proof

- By definition of the t-norm, we have
$$d(\varepsilon) = f_\&(d_1^{\varepsilon_1}, \ldots, d_N^{\varepsilon_N}) = f^{-1}(f(d_1^{\varepsilon_1}) \cdot \ldots \cdot f(d_N^{\varepsilon_N})).$$
- So, $d(\varepsilon) = f_\&(d_1^{\varepsilon_1}, \ldots, d_N^{\varepsilon_N}) = f^{-1}(e_1^{\varepsilon_1} \cdot \ldots \cdot e_N^{\varepsilon_N})$, where we denoted $e_i^{\varepsilon_i} \stackrel{\text{def}}{=} f(d_i^{\varepsilon_i})$.
- Since $f(x)$ is increasing, maximizing $d(\varepsilon)$ is equivalent to maximizing $e(\varepsilon) \stackrel{\text{def}}{=} f(d(\varepsilon)) = e_1^{\varepsilon_1} \cdot \ldots \cdot e_N^{\varepsilon_N}$.
- We required that the sequences $d^=$ and $d^{\neq}$ properly describe reasonableness.
- Thus, for each i, we have $d(\varepsilon_=) > d(\varepsilon_=^{(i)})$, where
$$\varepsilon_=^{(i)} \stackrel{\text{def}}{=} (=, \ldots, =, \neq \text{ (on } i\text{-th place)}, =, \ldots, =).$$
- This inequality is equivalent to $e(\varepsilon_=) > e(\varepsilon_=^{(i)})$.
- Since the values $e(\varepsilon)$ are simply the products, we thus conclude that $e_i^= > e_i^{\neq}$.

17. Proof (cont-d)

- Maximizing $e(\varepsilon)=\prod\limits_{i=1}^N e_i^{\varepsilon_i}$ is equivalent to maximizing $\dfrac{e(\varepsilon)}{c}$, for a constant $c\stackrel{\text{def}}{=}\prod\limits_{i=1}^N e_i^{=}$.
- The ratio $\dfrac{e(\varepsilon)}{c}$ can be reformulated as $\dfrac{e(\varepsilon)}{c}=\prod\limits_{i:\varepsilon_i=\neq}\dfrac{e_i^{\neq}}{e_i^{=}}$.
- Since $\ln(x)$ is increasing, maximizing this product is equivalent to minimizing minus logarithm

$$L(\varepsilon)\stackrel{\text{def}}{=}-\ln\left(\frac{e(\varepsilon)}{c}\right)=\sum_{i:\varepsilon_i=\neq} w_i,\ \text{ where } w_i\stackrel{\text{def}}{=}-\ln\left(\frac{e_i^{\neq}}{e_i^{=}}\right).$$

- Since $e_i^{=}>e_i^{\neq}>0$, we have $\dfrac{e_i^{\neq}}{e_i^{=}}<1$ and thus, $w_i>0$.
- The proposition is proven.

18. A Similar Derivation Can Be Obtained in the Probabilistic Case

- Reasonableness can be described by assigning a *probability* $p(\varepsilon)$ to each possible sequence ε.
- Let $p_i^=$ be the probability that $a_i = 0$, and let $p_i^{\neq} = 1 - p_i^=$ be the probability that $a_i \neq 0$.
- We do not know the relation between the values ε_i and ε_j corresponding to different coefficients $i \neq j$.
- So, it makes sense to assume that the corresponding random variables ε_i and ε_j are independent, so

$$p(\varepsilon) = \prod_{i=1}^{N} p_i^{\varepsilon_i}.$$

- So, we arrive at the following definitions.

19. Probabilistic Case: Definitions

- Let $p^= = (p^=_1, \ldots, p^=_N)$ be a sequence of real numbers from the interval $[0, 1]$, and let $p^{\neq}_i \stackrel{\text{def}}{=} 1 - p^=_i$.
- For each sequence ε, its *probability* is $p(\varepsilon) \stackrel{\text{def}}{=} \prod\limits_{i=1}^{N} p_i^{\varepsilon_i}$.
- We say that the sequence $p^=$ *properly describes reasonableness* if the following two conditions are satisfied:
  - the sequence $\varepsilon_= \stackrel{\text{def}}{=} (=, \ldots, =)$ is more probable than all others, i.e., $p(\varepsilon_=) > p(\varepsilon)$ for all $\varepsilon \neq \varepsilon_=$,
  - the sequence $\varepsilon_{\neq} \stackrel{\text{def}}{=} (\neq, \ldots, \neq)$ is less probable than all others, i.e., $p(\varepsilon_{\neq}) < p(\varepsilon)$ for all $\varepsilon \neq \varepsilon_{\neq}$.
- For each set S of sequences, we say that a sequence $\varepsilon \in S$ *is the most probable* if $p(\varepsilon) = \max\limits_{\varepsilon' \in S} p(\varepsilon')$.

20. Probabilistic Case: Main Result

- **Proposition.**
  - *Let us assume that the sequence* $p^=$ *properly describes reasonableness.*
  - *Then, there exist weights* $w_i > 0$ *for which, for each set* S*, the following two conditions are equivalent:*
    - *the sequence* $\varepsilon \in S$ *is the most probable,*
    - *the sum* $\sum\limits_{i:\varepsilon_i = \neq} w_i$ *is the smallest possible.*
- **Discussion.** In other words, probabilistic techniques also lead to the sparsity condition.

21. Fuzzy Approach vs. Probabilistic Approach

- *Fact:* the probabilistic approach leads to the same conclusion as the fuzzy approach.
- *First conclusion:* this makes us more confident that our justification of sparsity is valid.
- *Observation:*
  - the probability-based result is based on the assumption of independence, while
  - the fuzzy-based result can allow different types of dependence – as described by different t-norms.
- *Second conclusion:* this is an important advantage of the fuzzy-based approach.

Part II

Why ℓ_p-methods in Signal and Image Processing: A Fuzzy-Based Explanation

22. Need for Deblurring

- Cameras and other image-capturing devices are getting better and better every day.
- However, none of them is perfect, there is always some blur, that comes from the fact that:
  - while we would like to capture the intensity $I(x, y)$ at each spatial location (x, y),
  - the signal $s(x, y)$ is influenced also by the intensities $I(x', y')$ at nearby locations (x', y'):

$$s(x,y) = \int w(x,y,x',y') \cdot I(x',y')\,dx'\,dy'.$$

- When we take a photo of a friend, this blur is barely visible – and does not constitute a serious problem.
- However, when a spaceship takes a photo of a distant planet, the blur is very visible – so deblurring is needed.

23. In General, Signal and Image Reconstruction Are Ill-Posed Problems

- The image reconstruction problem is *ill-posed* in the sense that:
  - large changes in $I(x, y)$
  - can lead to very small changes in $s(x, y)$.
- Indeed, the measured value $s(x, y)$ is an average intensity over some small region.
- Averaging eliminates high-frequency components.
- Thus, for $I^*(x, y) = I(x, y) + c \cdot \sin(\omega_x \cdot x + \omega_y \cdot y)$, the signal is practically the same: $s^*(x, y) \approx s(x, y)$.
- However, the original images, for large c, may be very different.

24. Need for Regularization

- To reconstruct the image reasonably uniquely, we must impose additional conditions on the original image.
- This imposition is known as *regularization*.
- Often, a signal or an image is smooth (differentiable).
- Then, a natural idea is to require that the vector $d = (d_1, d_2, \ldots)$ formed by the derivatives is close to 0:

$$\rho(d, 0) \le C \Leftrightarrow \sum_{i=1}^{n} d_i^2 \le c \stackrel{\text{def}}{=} C^2.$$

- For continuous signals, sum turns into an integral:

$$\int (\dot{x}(t))^2\, dt \le c \text{ or } \int \left(\left(\frac{\partial I}{\partial x} \right)^2 + \left(\frac{\partial I}{\partial y} \right)^2 \right) dx\, dy \le c.$$

25. Tikhonov Regularization

- Out of all smooth signals or images, we want to find the best fit with observation: $J \stackrel{\text{def}}{=} \sum_i e_i^2 \to \min$.
- Here, e_i is the difference between the actual and the reconstructed values.
- Thus, we need to minimize J under the constraint

$$\int (\dot{x}(t))^2\,dt \le c \text{ and } \int \left(\left(\frac{\partial I}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y}\right)^2 \right) dx\,dy \le c.$$

- Lagrange multiplier method reduced this constraint optimization problem to the unconstrained one:

$$J + \lambda \cdot \int \left(\left(\frac{\partial I}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y}\right)^2 \right) dx\,dy \to \min_{I(x,y)}.$$

- This idea is known as *Tikhonov regularization*.

26. From Continuous to Discrete Images

- In practice, we only observe an image with a certain spatial resolution.
- So we can only reconstruct the values $I_{ij} = I(x_i, y_j)$ on a certain grid $x_i = x_0 + i \cdot \Delta x$ and $y_j = y_0 + j \cdot \Delta y$.
- In this discrete case, instead of the derivatives, we have differences:

$$J + \lambda \cdot \sum_i \sum_j ((\Delta_x I_{ij})^2 + (\Delta_y I_{ij})^2) \to \min_{I_{ij}} .$$

- Here:
  - $\Delta_x I_{ij} \stackrel{\text{def}}{=} I_{ij} - I_{i-1,j}$, and
  - $\Delta_y I_{ij} \stackrel{\text{def}}{=} I_{ij} - I_{i,j-1}$.

27. Limitations of Tikhonov Regularization and ℓ^p-Method

- Tikhonov regularization is based on the assumption that the signal or the image is smooth.
- In real life, images are, in general, not smooth.
- For example, many of them exhibit a fractal behavior.
- In such non-smooth situations, Tikhonov regularization does not work so well.
- To take into account non-smoothness, researchers have proposed to modify the Tikhonov regularization:
  - instead of the squares of the derivatives,
  - use the p-th powers for some $p \neq 2$:

$$J + \lambda \cdot \sum_i \sum_j (|\Delta_x I_{ij}|^p + |\Delta_y I_{ij}|^p) \to \min_{I_{ij}}.$$

- This works much better than Tikhonov regularization.

28. Remaining Problem

- *Problem:* the ℓ^p-methods are heuristic.
- There is no convincing explanation of why necessarily we replace the square:
  - with a p-th power and
  - not, for example, with some other function.
- *We show:* that a natural formalization of the corresponding intuitive ideas indeed leads to ℓ^p-methods.
- To formalize the intuitive ideas behind image reconstruction, we use *fuzzy techniques*.
- Fuzzy techniques were designed to transform:
  - imprecise intuitive ideas into
  - exact formulas.

29. Let Us Apply Fuzzy Techniques

- We are trying to formalize the statement that the image is continuous.
- This means that the differences $\Delta x_k \stackrel{\text{def}}{=} \Delta_x I_{ij}$ and $\Delta_y I_{ij}$ between image intensities at nearby points are small.
- Let $\mu(x)$ denote the degree to which x is small, and $f_\&(a,b)$ denote the "and"-operation.
- Then, the degree d to which Δx_1 is small *and* Δx_2 is small, etc., is:

$$d = f_\&(\mu(\Delta x_1), \mu(\Delta x_2), \mu(\Delta x_3), \ldots).$$

- *Known:* each "and"-operation can be approximated, for any $\varepsilon > 0$, by an *Archimedean* one:

$$f_\&(a,b) = f^{-1}(f(a)) \cdot f(b)).$$

- Thus, without losing generality, we can safely assume that the actual "and"-operation is Archimedean.

30. Analysis of the Problem

- We want to select an image with the largest degree of satisfying this condition:

$$d = f^{-1}(f(\mu(\Delta x_1))\cdot f(\mu(\Delta x_2))\cdot f(\mu(\Delta x_3))\cdot \ldots) \to \max .$$

- Since the function $f(x)$ is increasing, maximizing d is equivalent to maximizing

$$f(d) = f(\mu(\Delta x_1)) \cdot f(\mu(\Delta x_2)) \cdot f(\mu(\Delta x_3)) \cdot \ldots$$

- Maximizing this product is equivalent to minimizing its negative logarithm

$$L \stackrel{\text{def}}{=} -\ln(d) = \sum_k g(\Delta x_k), \text{ where } g(x) \stackrel{\text{def}}{=} -\ln(f(\mu(x))).$$

- In these terms, selecting a membership function is equivalent to selecting the related function $g(x)$.

31. Which Function $g(x)$ Should We Select: Idea

- The value $\Delta x_i = 0$ is small, so $\mu(0) = 1$ and $g(0) = -\ln(1) = 0$.
- The numerical value of a difference Δx_i depends on the choice of a measuring unit.
- If we choose a measuring unit (MU) which is a times smaller, then $\Delta x_i \to a \cdot \Delta x_i$.
- It's reasonable to request that the requirement $\sum\limits_k g(\Delta x_k) \to \min$ not change if we change MU.
- For example, if $g(z_1) + g(z_2) = g(z'_1) + g(z'_2)$, then

$$g(a \cdot z_1) + g(a \cdot z_2) = g(a \cdot z'_1) + g(a \cdot z'_2).$$

32. Main Result

- *Reminder:* selecting the most reasonable values of Δx_k ($d \to \max$) is equivalent to $\sum\limits_k g(\Delta x_k) \to \min$.
- *Main condition:* we are looking for a function $g(x)$ for which $g(z_1) + g(z_2) = g(z'_1) + g(z'_2)$, then

$$g(a \cdot z_1) + g(a \cdot z_2) = g(a \cdot z'_1) + g(a \cdot z'_2).$$

- *Main result:* $g(a) = C \cdot a^p + \text{const}$, for some $p > 0$.
- *Fact:* minimizing $\sum\limits_k g(\Delta x_k)$ is equivalent to minimizing the sum $\sum\limits_k |\Delta x_k|^p$.
- *Fact:* minimizing $\sum\limits_k |\Delta x_k|^p$ under condition $J \le c$ is equivalent to minimizing $J + \lambda \cdot \sum\limits_k |\Delta x_k|^p$.
- *Conclusion:* fuzzy techniques indeed justify ℓ^p-method.

33. Proof

- We are looking for a function $g(x)$ for which $g(z_1) + g(z_2) = g(z_1') + g(z_2')$, then
$$g(a \cdot z_1) + g(a \cdot z_2) = g(a \cdot z_1') + g(a \cdot z_2').$$
- Let us consider the case when $z_1' = z_1 + \Delta z$ for a small Δz, and $z_2' = z_2 + k \cdot \Delta z + o(\Delta z)$ for an appropriate k.
- Here, $g(z_1 + \Delta z) = g(z_1) + g'(z_1) \cdot \Delta z + o(\Delta z)$, so $g'(z_1) + g'(z_2) \cdot k = 0$ and $k = -\dfrac{g'(z_1)}{g'(z_2)}$.
- The condition $g(a \cdot z_1) + g(a \cdot z_2) = g(a \cdot z_1') + g(a \cdot z_2')$ similarly takes the form $g'(a \cdot z_1) + g'(z_2) \cdot k = 0$, so
$$g'(a \cdot z_1) - g'(a \cdot z_2) \cdot \frac{g'(z_1)}{g'(z_2)} = 0.$$
- Thus, $\dfrac{g'(a \cdot z_1)}{g'(z_1)} = \dfrac{g'(a \cdot z_2)}{g'(z_2)}$ for all a, z_1, and z_2.

34. Proof (cont-d)

- *Reminder:* $\dfrac{g'(a\cdot z_1)}{g'(z_1)}=\dfrac{g'(a\cdot z_2)}{g'(z_2)}$ for all z_1 and z_2.
- This means that the ratio $\dfrac{g'(a\cdot z_1)}{g'(z_1)}$ does not depend on z_i: $\dfrac{g'(a\cdot z_1)}{g'(z_1)}=F(a)$ for some $F(a)$.
- For $a=a_1\cdot a_2$, we have
$$F(a)=\frac{g'(a\cdot z_1)}{g'(z_1)}=\frac{g'(a_1\cdot a_2\cdot z_1)}{g'(z_1)}=$$
$$\frac{g'(a_1\cdot(a_2\cdot z_1))}{g'(a_2\cdot z_1)}\cdot\frac{g'(a_2\cdot z_1)}{g'(z_1)}=F(a_1)\cdot F(a_2).$$
- So, $F(a_1\cdot a_2)=F(a_1)\cdot F(a_2)$, thus $F(a)=a^q$ for some real number q.
- $\dfrac{g'(a\cdot z_1)}{g'(z_1)}=F(a)$ becomes $g'(a\cdot z_1)=g'(z_1)\cdot a^q$.

35. Proof (final part)

- *Reminder:* we have $g'(a \cdot z_1) = g'(z_1) \cdot a^p$.
- For $z_1 = 1$, we get $g'(a) = C \cdot a^q$, where $C \stackrel{\text{def}}{=} g'(1)$.
- We could have $q = -1$ or $q \neq -1$.
- For $q = -1$, we get $g(a) = C \cdot \ln(a) + \text{const}$, which contradicts to $g(0) = 0$.
- Integrating, for $q \neq -1$, we get

$$g(a) = \frac{C}{q+1} \cdot a^{q+1} + \text{const}.$$

- The main result is proven.

Part III

How to Improve the Existing Semi-Heuristic Technique

36. Blind Image Deconvolution: Formulation of the Problem

- The measurement results y_k differ from the actual values x_k dues to additive noise and blurring:

$$y_k = \sum_i h_i \cdot x_{k-i} + n_k.$$

- From the mathematical viewpoint, y is a *convolution* of h and x: $y = h \star x$.
- Similarly, the observed image $y(i, j)$ differs from the ideal one $x(i, j)$ due to noise and blurring:

$$y(i, j) = \sum_{i'} \sum_{j'} h(i - i', j - j') \cdot x(i', j') + n(i, j).$$

- It is desirable to reconstruct the original signal or image, i.e., to perform *deconvolution*.

37. Ideal No-Noise Case

- In the ideal case, when noise $n(i,j)$ can be ignored, we can find $x(i,j)$ by solving a system of linear equations:

$$y(i,j)=\sum_{i'}\sum_{j'} h(i-i',j-j')\cdot x(i',j').$$

- However, already for 256×256 images, the matrix h is of size 65,536×65,536, with billions entries.
- Direct solution of such systems is not feasible.
- A more efficient idea is to use Fourier transforms, since $y=h\star x$ implies $Y(\omega)=H(\omega)\cdot X(\omega)$; hence:
  - we compute $Y(\omega)=\mathcal{F}(y)$;
  - we compute $X(\omega)=\dfrac{Y(\omega)}{H(\omega)}$, and
  - finally, we compute $x=\mathcal{F}^{-1}(X(\omega))$.

38. Deconvolution in the Presence of Noise with Known Characteristics

- Suppose that signal and noise are independent, and we know the power spectral densities

$$S_I(\omega) = \lim_{T\to\infty} E\left[\frac{1}{T}\cdot |X_T(\omega)|^2\right], S_N(\omega) = \lim_{T\to\infty} E\left[\frac{1}{T}\cdot |N_T(\omega)|^2\right].$$

- We minimize the expected mean square difference

$$d \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\cdot E\left[\int_{-T/2}^{T/2} (\widehat{x}(t) - x(t))^2\,dt\right].$$

- Minimizing d leads to the known Wiener filter formula

$$\widehat{X}(\omega_1,\omega_2) = \frac{H^*(\omega_1,\omega_2)}{|H(\omega_1,\omega_2)|^2 + \dfrac{S_N(\omega_1,\omega_2)}{S_I(\omega_1,\omega_2)}}\cdot Y(\omega_1,\omega_2).$$

39. Blind Image Deconvolution in the Presence of Prior Knowledge

- Wiener filter techniques assume that we know the blurring function h.
- In practice, we often only have partial information about h.
- Such situations are known as *blind deconvolution*.
- Sometimes, we know a joint probability distribution $p(\Omega, x, h, y)$ corresponding to some parameters Ω:

$$p(\Omega, x, h, y) = p(\Omega) \cdot p(x|\Omega) \cdot p(h|\Omega) \cdot p(y|x, h, \Omega).$$

- In this case, we can find

$$\widehat{\Omega} = \arg\max_{\Omega} p(\Omega|y) = \int\int_{x,h} p(\Omega, x, h, y)\, dx\, dh \text{ and}$$

$$(\widehat{x}, \widehat{h}) = \arg\max_{x,h} p(x, h|\widehat{\Omega}, y).$$

40. Blind Image Deconvolution in the Absence of Prior Knowledge: Sparsity-Based Techniques

- In many practical situations, we do not have prior knowledge about the blurring function h.
- Often, what helps is *sparsity* assumption: that in the expansion $x(t) = \sum_i a_i \cdot e_i(x)$, most a_i are zero.
- In this case, it makes sense to look for a solution with the smallest value of

$$\|a\|_0 \stackrel{\text{def}}{=} \#\{i : a_i \neq 0\}.$$

- The function $\|a\|_0$ is not convex and thus, difficult to optimize.
- It is therefore replaced by a close *convex* objective function $\|a\|_1 \stackrel{\text{def}}{=} \sum_i |a_i|$.

41. State-of-the-Art Technique for Sparsity-Based Blind Deconvolution

- Sparsity is the main idea behind the algorithm described in (Amizic et al. 2013) that minimizes

$$\frac{\beta}{2}\cdot\|y-\mathbf{W}a\|_2^2+\frac{\eta}{2}\cdot\|\mathbf{W}a-\mathbf{H}x\|_2^2+\tau\cdot\|a\|_1+\alpha\cdot R_1(x)+\gamma\cdot R_2(h).$$

- Here, $R_1(x) = \sum\limits_{d\in D} 2^{1-o(d)} \sum_i |\Delta_i^p(x)|^p$, where $\Delta_i^p(x)$ is the difference operator, and
- $R_2(h) = \|\mathbf{C}h\|^2$, where $\mathbf{C}$ is the discrete Laplace operator.
- The ℓ^p-sum $\sum\limits_i |v_i(x)|^p$ is optimized as $\sum\limits_i \frac{(v_i(x^{(k)}))^2}{v_i^{2-p}}$, where $v_i = v_i(x^{(k-1)})$ for x from the previous iteration.
- This method results in the best blind image deconvolution.

42. Need for Improvement

- The current technique is based on minimizing the sum $|\Delta_x I|^p + |\Delta_y I|^p$.
- This is a discrete analog of the term $\left|\dfrac{\partial I}{\partial x}\right|^p + \left|\dfrac{\partial I}{\partial y}\right|^p$.
- For $p = 2$, this is the square of the length of the gradient vector and is, thus, rotation-invariant.
- However, for $p \neq 2$, the above expression is not rotation-invariant.
- Thus, even if it works for some image, it may not work well if we rotate this image.
- To improve the quality of image deconvolution, it is thus desirable to make the method rotation-invariant.
- We show that this indeed improves the quality of deconvolution.

43. Rotation-Invariant Modification: Description and Results

- We want to replace the expression $\left|\dfrac{\partial I}{\partial x}\right|^p + \left|\dfrac{\partial I}{\partial y}\right|^p$ with a rotation-invariant function of the gradient.
- The only rotation-invariant characteristic of a vector a is its length $\|a\| = \sqrt{\sum\limits_i a_i^2}$.
- Thus, we replace the above expression with

$$\left(\left|\frac{\partial I}{\partial x}\right|^2 + \left|\frac{\partial I}{\partial y}\right|^2\right)^{p/2}.$$

- Its discrete analog is $((\Delta_x I)^2 + (\Delta_y I)^2)^{p/2}$.
- This modification leads to a statistically significant improvement in reconstruction accuracy $\|\widehat{x} - x\|_2$.

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