Practical Need for Algebraic (Equality-Type) Solutions of Interval Equations and for Extended-Zero Solutions

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1. Need for Data Processing

- We are often interested in the values of quantities $y_1, \ldots, y_m$ which are difficult to measure directly.
- *Examples:* distance to a faraway star, tomorrow's temperature at a certain location.
- Since we cannot measure these quantities directly, to estimate these quantities we must:
  - find easier-to-measure quantities $x_1, \ldots, x_n$ which are related to y_i by known formulas $y_i = f_i(x_1, \ldots, x_n)$,
  - measure these quantities x_j, and
  - use the results $\widetilde{x}_j$ of measuring the quantities x_j to compute the estimates for y_i:

$$\widetilde{y}_i = f(\widetilde{x}_1, \ldots, \widetilde{x}_n).$$

- Computation of these estimates is called *indirect measurement* or *data processing.*

2. Need for Data Processing under Uncertainty

- Measurements are never 100% accurate.
- Hence, the measurement result $\widetilde{x}_j$ is, in general, different from the actual (unknown) value x_j.
- In other words, the measurement errors $\Delta x_j \stackrel{\text{def}}{=} \widetilde{x}_j - x_j$ are, in general, different from 0.
- Because of this, the estimates $\widetilde{y}_i$ are, in general, different from the desired values y_i.
- It is therefore desirable to know how accurate are the resulting estimates.

3. Need for Interval Uncertainty

- The manufacturer of the measuring instrument usually provides a bound Δ_j on the measurement error:

$$|\Delta x_j| \le \Delta_j.$$

- If no such bound is known, this is not a measuring instrument, but a wild-guess-generator.
- Sometimes, we also know the probabilities of different values Δx_j within this interval.
- However, in many practical situations, the upper bound is the only information that we have; then:
  - after we know the result $\widetilde{x}_j$ of measuring x_j,
  - the only information that we have about the actual (unknown) value x_j is that $x_j \in [\underline{x}_j, \overline{x}_j]$, where:

$$\underline{x}_j \stackrel{\text{def}}{=} \widetilde{x}_j - \Delta_j \text{ and } \overline{x}_j \stackrel{\text{def}}{=} \widetilde{x}_j + \Delta_j.$$

4. Need for Interval Computations

- In this case, all we know about each x_i is that

$$x_i \in [\underline{x}_i, \overline{x}_i].$$

- We also know that $y_i = f_i(x_1, \ldots, x_n)$.
- Then, the only thing that we can say about each value $y_i = f_i(x_1, \ldots, x_n)$ is that y_i is in the range

$$\{f_i(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

- Computation of this range is one of the main problems of *interval computations*.

5. Sometimes, We do not Know the Exact Dependence

- So far, we assumed that when we know the exact dependence $y_i = f_i(x_1, \ldots, x_n)$ between y_i and x_j.
- In practice, often, we do not know the exact dependence.
- Instead, we know that the dependence belongs to a finite-parametric *family* of dependencies, i.e., that

$$y_i = f_i(x_1, \ldots, x_n, a_1, \ldots, a_k) \text{ for some parameters } a_1, \ldots, a_k.$$

- *Example:* y_i is a linear function of x_j, i.e.,

  $y_i = c_i + \sum_{j=1}^{n} c_{ij} \cdot x_j$ for some c_i and c_{ij}.
- The presence of these parameters complicates the corresponding data processing problem.
- Depending on what we know about the parameters, we have different situations.

6. Specific Case: Control Solution

- Sometimes, we can *control* the values a_ℓ, by setting them to any values within certain intervals $[\underline{a}_\ell, \overline{a}_\ell]$.
- By setting the appropriate values of the parameters, we can change the values y_i.
- We would like the values y_i to be within some given ranges $[\underline{y}_i, \overline{y}_i]$.
- For example, we would like the temperature to be within a comfort zone.
- So, we need to find x_j for which, by applying controls $a_i \in [\underline{a}_\ell, \overline{a}_\ell]$, we can place each y_i within $[\underline{y}_i, \overline{y}_i]$:

$$X = \{x : \exists a_\ell \in [\underline{a}_\ell, \overline{a}_\ell]\, \forall i\, f_i(x_1, \ldots, x_n, a_1, \ldots, a_k) \in [\underline{y}_i, \overline{y}_i]\}.$$

- This set is known as the *control solution* to the corresponding interval system of equations $f(x, a) = y$.

7. Situation When We Need to Find the Parameters from the Data

- Sometimes, we do not know these values a_ℓ, we must determine these values from the measurements.
- After each cycle c of measurements, we conclude that:
  - the actual (unknown) value of $x_j^{(c)}$ is in the interval $[\underline{x}_j^{(c)}, \overline{x}_j^{(c)}]$ and
  - the actual value of $y_i^{(c)}$ is in the interval $[\underline{y}_i^{(c)}, \overline{y}_i^{(c)}]$.
- We want to find the set A of all the values a for which $y^{(c)} = f(x^{(c)}, a)$ for some $x^{(c)}$ and $y^{(c)}$:

$$A = \{a : \forall c\, \exists x_j^{(c)} \in [\underline{x}_j^{(c)}, \overline{x}_j^{(c)}]\, \exists y_i^{(c)} \in [\underline{y}_i^{(c)}, \overline{y}_i^{(c)}]\, (f(x^{(c)}, a) = y^{(c)})\}.$$

- This set A is known as the *united solution* to the interval system of equations.

8. Comment About Notations

- In general, in our description:
  - y denotes the desired quantities,
  - x denote easier-to-measure quantities, and
  - a denote parameters of the dependence between these quantities.
- In some cases, we have some information about a, and we need to know x – case of the control solution.
- In other cases, we have some information about x, and we need to know a – case of the united solution.
- As a result, sometimes x's are the unknowns, and sometimes a's are the unknowns.

9. What Can We Do Once We Have Found the Range of Possible Values of a

- Once we have found the set A of possible values of a, we can find the range of possible values of y_i:

$$\{f_i(x_1, \ldots, x_n, a) : x_j \in [\underline{x}_j, \overline{x}_j] \text{ and } a \in A\}.$$

- This is a particular case of the main problem of interval computations.
- Often, we want to make sure that each value y_i lies within the given bounds $[\underline{y}_i, \overline{y}_i]$.
- Then we must find the set X of possible values of x for which $f_i(x, a) \in [\underline{y}_i, \overline{y}_i]$ for all $a \in A$:

$$X = \{x : \forall a \in A \, \forall i \, (f_i(x, a) \in [\underline{y}_i, \overline{y}_i])\}.$$

- This set is known as the *tolerance solution* to the interval system of equations.

10. Sometimes, the Values a May Change

- Up to now, we consider the cases when the values a_ℓ are either fixed, or can be changed by us.
- In practice, these values may change in an unpredictable way.
- For example, these parameters may represent some physical processes that influence y_i's.
- We therefore do not know the exact values of a_ℓ, only the bounds $[\underline{a}_\ell, \overline{a}_\ell]$.
- So, the set A of all possible combinations $a = (a_1, \ldots, a_k)$ is contained in a box:

$$A \subseteq [\underline{a}_1, \overline{a}_1] \times \ldots \times [\underline{a}_k, \overline{a}_k].$$

- For example, the set A can be an ellipsoid.

11. Sometimes, the Values a May Change (cont-d)

- In this case, we can still solve the same two problems whose solutions we described earlier.
- We can solve the main problem of interval computations – the problem of computing the range.
- This way we find the set Y of possible values of y.
- We can also solve the corresponding tolerance problem.
- This way, we find the set of values x that guarantee that each y_i is within the desired interval.

12. Is This All There Is?

- There are also more complex problems.
- However, most interval computation packages support the above four problems:
  - range estimation,
  - finding a control solution,
  - finding a united solution, and
  - finding a tolerance solution.
- We show: in practice, we need to use a different notion of an *algebraic* (equality-type) solution.
- This notion:
  - has been previously proposed and analyzed
  - but is not usually included in interval computations packages.

13. How to Find the Set A?

- We considered the case when the values of the parameter a can change.
- We assumed that we know the set A of possible values of the corresponding parameter vector a.
- But how do we find this set?
- All information comes from measurements.
- The only relation between the parameters a and measurable quantities is the formula $y = f(x, a)$.
- Thus, to find the set A of possible values of a, we need to measure x and y many times; so, we get:
  - the set X of possible values of the vector x and
  - the set Y of possible values of the vector y.
- Based on the sets X and Y, we need to find the A.

14. Independence

- It is reasonable to assume that x and a are *independent* in some reasonable sense.
- Independence notion is well known for probabilities: the probability of x does not depend on a:

$$P(x \,|\, a) = P(x \,|\, a') \text{ for all } a, a'.$$

- In the interval case, we do not know the probabilities, we only know which pairs (x, a) are possible.
- We have a *set* $S \subseteq X \times A$ of possible pairs (x, a).
- So, we arrive at the following definition:
- x and a are *independent* if the set $S_a = \{x : (x, a) \in S\}$ of possible values of x does not depend on a: $S_a = S_{a'}$.

15. What We Can Now Conclude About the Dependence Between A, X, and Y

- **Proposition.** *x and a are independent if and only if S is a Cartesian product, i.e.,*

$$S = s_x \times s_a \textit{ for some } s_x \subseteq X \textit{ and } s_a \subseteq A.$$

- Thus, the set Y is equal to the range of $f(x, a)$ when $x \in X$ and $a \in A$.
- So, we look for sets A for which

$$Y = f(X, A) \stackrel{\text{def}}{=} \{f(x, a) : x \in X \text{ and } a \in A\}.$$

- This set A is known as an *algebraic* (*formal*, *equality-type*) solution to the interval system of equations.
- This notion was introduced and studied by Nickel, Ratschek, Shary, et al.

16. What If There Is No Algebraic Solution

- Sometimes, the corresponding problem has no solutions.
- For example, for $f(x, a) = x + a$, with $Y = [-1, 1]$ and $X = [-2, 2]$, there is no solution.
- The width $w(X + A)$ of $X + A$ is always $\geq w(X) = 4$ of X and thus, cannot be equal to $w(Y) = 2$.
- What shall we do in this case?
- Of course, this would not happen if we had the *actual* ranges X and Y.
- So, the fact that we cannot find A means something is wrong with these estimates.
- To find out what can be wrong, let us recall how the ranges can be obtained from the experiments.

17. How Ranges Can Be Obtained From Experiments?

- For example, in the 1-D case, we perform several measurements of the quantity x_1 in different situations.
- Based on the measurement results $x_1^{(c)}$, we conclude that the set of possible values includes

$$[\underline{x}_1^{\approx}, \overline{x}_1^{\approx}], \text{ where } \underline{x}_1^{\approx} \stackrel{\text{def}}{=} \min_c x_1^{(c)} \text{ and } \overline{x}_1^{\approx} \stackrel{\text{def}}{=} \max_c x_1^{(c)}.$$

- Of course, we can also have some values outside this interval.
- Example: for a uniform distribution on $[0,1]$, the interval $[\underline{x}^{\approx}, \overline{x}^{\approx}]$ is narrower than $[0,1]$.
- The fewer measurement we take, the narrower this interval.
- So, to estimate the actual range, we *inflate* the interval $[\underline{x}_1^{\approx}, \overline{x}_1^{\approx}]$.

18. Back to Our Problem: What If There Is No Formal Solution

- That we have a mismatch between X and Y means that one of the intervals was not inflated enough.
- X corresponds to easier-to-measure quantities.
- We can thus measure x many times.
- So, even without inflation, get pretty accurate estimates of the actual range X.
- On the other hand, the values y are difficult to measure.
- For these values, we do not have as many measurement results and thus, there is a need for inflation.
- So, we can safely assume that the range for X is reasonably accurate, but the range of Y needs inflation.
- To make this idea precise, let us formalize what is an inflation.

19. What Is an Inflation: Analysis of the Problem

- We want to define a mapping I that transforms each non-degenerate interval $\mathbf{x} = [\underline{x}, \overline{x}]$ into a wider interval

$$I(\mathbf{x}) \supset \mathbf{x}.$$

- What are the natural properties of this transformation?
- The numerical value x of the corresponding quantity depends:
  - on the choice of the measuring unit,
  - on the choice of the starting point, and
  - sometimes, on the choice of direction.
- Example: we can measure temperature t_C in Celsius,
- We can also use a different measuring unit and a different starting point, and get $t_F = 1.8 \cdot t_C + 32$.

20. What Is an Inflation (cont-d)

- We can use the usual convention and consider the usual signs of the electric charge.
- We could also use the opposite signs – then an electron would be a positive electric charge.
- It is reasonable to require that the result of the inflation transformation does not change if we simply:
  - change the measuring units,
  - change the starting point, and/or
  - change the sign.
- Changing the starting point leads to a new interval $[\underline{x},\overline{x}]+x_0=[\underline{x}+x_0,\overline{x}+x_0]$ for some x_0.
- Changing the measuring unit leads to $\lambda\cdot[\underline{x},\overline{x}]=[\lambda\cdot\underline{x},\overline{x}]$ for some $\lambda>0$.
- Changing the sign leads to $-[\underline{x},\overline{x}]=[-\overline{x},-\underline{x}]$.

21. What Is an Inflation: Resulting Definition and the Main Result

- So, an *inflation* is a mapping from non-degenerate intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ to $I(\mathbf{x}) \supseteq \mathbf{x}$ such that:
  - for every x_0, we have $I(\mathbf{x} + x_0) = I(\mathbf{x}) + x_0$;
  - for every $\lambda > 0$, we have $I(\lambda \cdot \mathbf{x}) = \lambda \cdot I(\mathbf{x})$; and
  - we have $I(-\mathbf{x}) = -I(\mathbf{x})$.
- **Proposition.** *Every inflation operation has the form*

  $[\widetilde{x} - \Delta, \widetilde{x} + \Delta] \to [\widetilde{x} - \alpha \cdot \Delta, \widetilde{x} + \alpha \cdot \Delta]$ *for some* $\alpha > 1$.
- So how do we find A?
- We want to make sure that $f(X, A)$ is equal to the result of a proper inflation of Y.
- How can we tell that an interval Y' is the result of a proper inflation of Y?

22. So How Do We Find A?

- How can we tell that an interval Y' is the result of a proper inflation of Y?
- One can check that this is equivalent to the fact that the difference $Y' - Y$ is a symmetric interval $[-u, u]$.
- Such intervals are known as *extended zeros*; thus:
  - if we cannot find the set A for which $Y = f(X, A)$,
  - we should look for the set A for which the difference $f(X, A) - Y$ is an extended zero.
- What if we have several variables, i.e., $m > 1$?
- In this case, we may have different inflations for different components Y_i of the set Y.
- So, we should look for the set A for which, for all i, the difference $f_i(X, A) - Y_i$ is an extended zero.

23. Acknowledgments

This work was supported in part:

- by the grant DEC-2013/11/B/ST6/00960 from the National Science Center (Poland),
- by the US National Science Foundation grants HRD-0734825, HRD-1242122, and DUE-0926721, and
- by an award from Prudential Foundation.

24. Appendix 1: Proof of the Independence Result

- **Proposition.** *x and a are independent if and only if S is a Cartesian product, i.e.,*

$$S = s_x \times s_a \text{ for some } s_x \subseteq X \text{ and } s_a \subseteq A.$$

- If $S = s_x \times s_a$, then $S_a = s_x$ for each a and thus, $S_a = S_{a'}$ for all $a, a' \in A$.
- Vice versa, let us assume that x and a are independent.
- Let us denote the common set $S_a = S_{a'}$ by s_x.
- Let us denote by s_a, the set of all possible values a, i.e., the set of all a for which $(x, a) \in S$ for some x.
- Let us prove that in this case, $S = s_x \times s_a$.
- Indeed, if $(x, a) \in S$, then, by definition of s_x, $x \in S_a = s_x$, and, by definition of s_a, $a \in s_a$.

25. Proof of the Independence Result (cont-d)

- We have shown that $x \in s_x$ and $a \in s_a$.
- Thus, by the definition of the Cartesian product $B \times C$ as the set of all pairs (b, c), $b \in B$, $c \in C$, we have

$$(x, a) \in s_x \times s_a.$$

- Vice versa, let $(x, a) \in s_x \times s_a$, i.e., let $x \in s_x$ and $a \in s_a$.
- By definition of the set s_x, we have $S_a = s_x$, thus $x \in S_a$.
- By definition of the set S_a, this means that $(x, a) \in S$.
- The proposition is proven.

26. Appendix 2: Proof of the Inflation Result

- **Proposition.** *Every inflation operation has the form* $[\widetilde{x}-\Delta,\widetilde{x}+\Delta]\to[\widetilde{x}-\alpha\cdot\Delta,\widetilde{x}+\alpha\cdot\Delta]$ *for some* $\alpha>1$.
- It is easy to see that the above operation satisfies all the properties of an inflation.
- Let us prove that, vice versa, every inflation has this form.
- Indeed, for intervals $\mathbf{x}$ of type $[-\Delta,\Delta]$, we have $-\mathbf{x}=\mathbf{x}$, thus $I(\mathbf{x})=I(-\mathbf{x})$.
- On the other hand, due to the sign-invariance, we should have $I(-\mathbf{x})=-I(\mathbf{x})$.
- Thus, for the interval $[\underline{v},\overline{v}]\stackrel{\text{def}}{=}I(\mathbf{x})$, we should have $-[\underline{v},\overline{v}]=[-\overline{v},-\underline{v}]=[\underline{v},\overline{v}]$ and thus, $\underline{v}=-\overline{v}$.
- So, we have $I([-\Delta,\Delta])=[-\Delta'(\Delta),\Delta'(\Delta)]$ for some Δ' depending on Δ

27. Proof of the Inflation Result (cont-d)

- Since we should have $[-\Delta, \Delta] \subset I([-\Delta, \Delta])$, we must have $\Delta'(\Delta) > \Delta$.
- Let us denote $\Delta'(1)$ by α.
- Then, $\alpha > 1$ and $I([-1, 1]) = [-\alpha, \alpha]$.
- By applying scale-invariance, with $\lambda = \Delta$, we can then conclude that

$$I([-\Delta, \Delta]) = [-\alpha \cdot \Delta, \alpha \cdot \Delta].$$

- By applying shift-invariance, with $x_0 = \widetilde{x}$, we get the desired equality

$$I([\widetilde{x} - \Delta, \widetilde{x} + \Delta]) = [\widetilde{x} - \alpha \cdot \Delta, \widetilde{x} + \alpha \cdot \Delta].$$

- The proposition is proven.