From Interval-Valued Probabilities to Interval-Valued Possibilities: Case Studies of Interval Computation under Constraints

L. Gutierrez[1], M. Ceberio[1], V. Kreinovich[1], R. L. Gruver[1], M. Peña[1], M. J. Rister[1], A. Saldaña[1], J. Vasquez[1], J. Ybarra[1], and S. Benferhat[2]

[1]Department of Computer Science, University of Texas at El Paso, El Paso, TX 79968, USA, lcgutierrez@miners.utep.edu, vladik@utep.edu

[2]Centre de Recherche en Informatique de Lens CRIL, Université d'Artois, F62307 Lens Cedex, France, benferhat@cril.univ-artois.fr

1. Introduction and Motivation

- In many real-life situations, we need to make decisions.
- We must decide whether a pattern of behavior represents an intrusion – and if yes, activate defenses.
- In medicine, we need to decide on the best way to cure a patient.
- Each decision is based on our knowledge about the situation. This knowledge comes from two sources.
- First, we have records of previous experiences: e.g., records of different patients with different symptoms.
- We also have knowledge of experts: a skilled medical doctor can distinguish allergy cough from cold.
- Let us describe how to represent and process both types of knowledge.

2. Empirical Knowledge: Reminder

- The available information is usually incomplete.
- Based on this partial information, we often cannot make a definite conclusion about the present.
- For example, repeated attempts to login:
  - may represent intrusion, and
  - may also mean that a user forgot which of his passwords works where, and tries them all.
- In the past, there may have been many different situations similar to what we observe now.
- Their detailed analysis showed that they have been caused by different phenomena.
- We can say how frequently phenomena of different type were encountered in similar past situations.

3. Empirical Knowledge Is Usually Represented in Terms of Probabilities

- We can say how frequently phenomena of different type were encountered in similar past situations.
- For example, we may know that in situations with repeated logins,
  - in 10% of the cases, repeated logins were done by absent-minded users,
  - in 90% of the cases, it was an intrusion attempt.
- In other words, based on our prior experience, we know the *probabilities* of different phenomena.
- This is all the information that we can immediately deduce from the past.
- Thus, a general way to describe empirical knowledge is by describing the corresponding probabilities.

4. Need to Process Probabilities

- It is known that consistent preferences under such uncertainty can be described as follows:
  - we find utilities $u_1, \ldots, u_n$ of different outcomes;
  - we select a decision a s.t. $\sum\limits_{i=1}^{n} p_i(a) \cdot u_i \to \max$.
- So, we need to be able to compute the corresponding *expected values*.
- To compute the expected value, we need to know the probabilities $p_i(a)$.
- Sometimes, we know, e.g., $p(\text{cough} \mid \text{allergy})$, and we want $p(\text{allergy} \mid \text{cough})$.
- To compute these probabilities, we can use Bayes formula – which is based on *conditional probabilities*.
- So, we need to compute conditional probabilities $p(A \mid B)$.

5. How to Represent Expert Knowledge?

- Sometimes, the expert can estimate the probabilities of different phenomena.
- However, often, an expert can only provide *partial information* about these probabilities.
- Often, an expert can only estimate *relative probabilities* $r_{ij} = p_i/p_j$, but not the actual values p_i.
- In this case, it is convenient to "normalize" these values so that max is 1, i.e., take $\mu_i = c \cdot p_i$, with $\max \mu_i = 1$.
- These normalized values μ_i are known as *possibilities*.
- Sometimes, the expert only knows the *order* between probabilities, i.e., knows the values μ_i such that

$$\mu_i > \mu_j \Leftrightarrow p_i > p_j.$$

- In this case, we can also normalize to 1: $\max \mu_i = 1$.

6. How to Compute Conditional Probabilities and Possibilities

- If we knew $p_1, \ldots, p_n$, and then learn that $i \in S \subset \{1, \ldots, n\}$, then $p(i \,|\, S) = 0$ for $i \notin S$, and
$$p(i \,|\, S) = \frac{p_i}{\sum\limits_{j \in S} p_j} \text{ if } i \in S.$$
- For possibilities $\mu_1, \ldots, \mu_n$, after normalization, we get $\mu(i \,|\, S) = 0$ for $i \notin S$, and
$$\mu(i \,|\, S) = \frac{\mu_i}{\max\limits_{j \in S} \mu_j} \text{ if } i \in S.$$
- For "qualitative" possibilities, it is sufficient to get the largest value to 1:
$$\mu(i \,|\, S) = \begin{cases} 1 & \text{if } i \in S \text{ and } \mu_i = \max\limits_{j \in S} \mu_j \\ \mu_i & \text{if } i \in S \text{ and } \mu_i < \max\limits_{j \in S} \mu_j \\ 0 & \text{if } i \notin S \end{cases}$$

7. Probabilities and Possibilities Are Often Only Known With Interval Uncertainty

- Probabilities and possibilities have been successfully applied – e.g., in intrusion detection.
- In practice, we usually only know the probabilities p_i and the possibilities μ_i with uncertainty.
- Often, we only know the bounds on each of these values, i.e., we know the *intervals* $\left[\underline{p}_i, \overline{p}_i\right]$ and $\left[\underline{\mu}_i, \overline{\mu}_i\right]$.
- We need to be able to compute the ranges of possible values of conditional probabilities and possibilities, etc.
- This is the problem that we will analyze in this talk.
- In many applications, we have a large amount of data to process, and need for real-time decisions.
- It is thus also important to make the range-computing algorithms as fast as possible.

8. Need to "Narrow" Intervals: An Important Auxiliary Problem

- Before we start processing, it is important to find out which values from the intervals are actually possible.
- For example, if $n = 2$ and $p_1 = 0.5$, then – even if $\left[\underline{p}_2, \overline{p}_2\right] = [0,1]$ – the only possible value of p_2 is 0.5.
- In general, we have a set of intervals $[\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]$ and a constraint $g(x_1, \ldots, x_n) = c$:
  - for probabilities, the constraint is $\sum\limits_{j=1}^{n} p_j = 1$;
  - for possibilities, the constraint is $\max\limits_j \mu_j = 1$.
- For each i, we want to compute the "narrowed" interval

$$\{x_i \in [\underline{x}_i, \overline{x}_i] : \exists x_1 \ldots \exists x_{i-1} \exists x_{i+1} \ldots \exists x_n$$

$$(x_1 \in [\underline{x}_1, \overline{x}_1] \,\&\, \ldots \,\&\, x_n \in [\underline{x}_n, \overline{x}_n] \,\&\, g(x_1, \ldots, x_n) = c)\}.$$

9. Narrowing Intervals: Case of Probabilities

- Since $p_i = 1 - \sum\limits_{j \neq i} p_j$, we have $p_i \geq 1 - \sum\limits_{j \neq i} \overline{p}_j$ and $p_i \leq 1 - \sum\limits_{j \neq i} \underline{p}_j$. Thus, the new bounds on p_i are

$$\left[\max\left(\underline{p}_i, 1 - \sum_{j \neq i} \overline{p}_j\right), \min\left(\overline{p}_i, 1 - \sum_{j \neq i} \underline{p}_j\right)\right].$$

- Straightforward computation requires $O(n)$ time for each i, i.e., $O(n^2)$.
- It is faster to first compute $\underline{P} = \sum\limits_{i=1}^{n} \underline{p}_i$ and $\overline{P} = \sum\limits_{i=1}^{n} \overline{p}_i$, then we need linear time $O(n)$ to compute

$$\left[\max\left(\underline{p}_i, 1 - \overline{P} + \overline{p}_i\right), \min\left(\overline{p}_i, 1 - \underline{P} + \underline{p}_i\right)\right].$$

- We cannot compute faster: we need to compute n intervals, so we cannot use fewer than $O(n)$ steps.

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 10 of 29
Go Back
Full Screen
Close
Quit

10. Narrowing Intervals: Case of Possibilities

- A sequence of possibility intervals $\left[\underline{\mu}_i, \overline{\mu}_i\right] \subseteq [0,1]$ is consistent if and only if $\max\limits_i \overline{\mu}_i = 1$.
- If this sequence is consistent, then the corresponding narrowed intervals have the following form.
- If we have several intervals with $\overline{\mu}_j = 1$, then there is no narrowing:

$$[\mu_i^-, \mu_i^+] = \left[\underline{\mu}_i, \overline{\mu}_i\right] \text{ for each } i.$$

- If there is only one interval with $\overline{\mu}_j = 1$, then:
  - for this interval, $[\mu_j^-, \mu_j^+] = [1,1]$, while
  - for all other intervals $i \neq j$, there is no narrowing:

$$[\mu_i^-, \mu_i^+] = \left[\underline{\mu}_i, \overline{\mu}_i\right].$$

11. Interval Computation under Constraints: A General Problem

- Traditional interval computations:
  - we know an algorithm $f(x_1, \ldots, x_n)$;
  - we know the intervals $[\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]$;
  - we want to compute the range

$$\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1] \,\&\, \ldots \,\&\, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

- Interval computation under constraints:
  - we know algorithms $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ and a number c;
  - we know the intervals $[\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]$;
  - we want to compute the range

$$\{f(x_1, \ldots, x_n) : \forall i\, (x_i \in [\underline{x}_i, \overline{x}_i]) \,\&\, g(x_1, \ldots, x_n) = c\}.$$

12. Are All Combinations Possible?

- Often, we estimate the values of several characteristics

$$y_1 = f_1(x_1, \ldots, x_n), \ldots, y_m = f_m(x_1, \ldots, x_n).$$

- Sometimes, there is an additional constraint relating these characteristics: $h(y_1, \ldots, y_m) = d$.
- In this case, interval computation under constraints enable us to find the ranges $[\underline{y}_1, \overline{y}_1]$, ..., $[\underline{y}_m, \overline{y}_m]$.
- A natural question is: are all combinations $(y_1, \ldots, y_m)$ with $y_j \in [\underline{y}_j, \overline{y}_j]$ with $h(y_1, \ldots, y_m) = d$ possible?
- In other words, can we find $x_i \in [\underline{x}_i, \overline{x}_i]$ for which $g(x_1, \ldots, x_n) = c$ and $y_j = f_j(x_1, \ldots, x_n)$?
- If all comb. are possible, then the set S of all combinations $(y_1, \ldots, y_m)$ is $S = B \stackrel{\text{def}}{=} [\underline{y}_1, \overline{y}_1] \times \ldots \times [\underline{y}_m, \overline{y}_m]$.
- Otherwise, S is a proper subset of the box B.

13. Conditional Probabilities $q_i \stackrel{\text{def}}{=} p(i\,|\,S)$: Formulation of the Problem

- We know the intervals $\left[\underline{p}_i, \overline{p}_i\right] \subseteq [0,1]$.
- We assume that these intervals have already been narrowed.
- We know the condition S, i.e., a set $S \subset \{1, \ldots, n\}$.
- We want to find, for each $i \in S$,
  - the range $\left[\underline{q}_i, \overline{q}_i\right]$ of possible values of the conditional probability

$$q_i = \frac{p_i}{\sum\limits_{j \in S} p_j}$$

  - when $p_j \in \left[\underline{p}_j, \overline{p}_j\right]$ and $\sum\limits_{j=1}^{n} p_j = 1$.

14. What If We Ignore the Constraint

- If we divide both the numerator and the denominator of $q_i = \dfrac{p_i}{\sum\limits_{j\in S} p_j}$ by p_i, we get

$$q_i = \frac{1}{1 + \sum\limits_{j\in S,\ j\neq i} \dfrac{p_j}{p_i}}.$$

- This expression is increasing in p_i and decreasing in all other values p_j, $j \neq i$; thus:
  - $q_i \to$ min when $p_i \to$ min and $p_j \to$ max for $j \neq i$;
  - $q_i \to$ max when $p_i \to$ max and $p_j \to$ min for $j \neq i$.
- So, we get the range $\left[\dfrac{\underline{p}_i}{\underline{p}_i + \sum\limits_{j\in S,\ j\neq i} \overline{p}_j}, \dfrac{\overline{p}_i}{\overline{p}_i + \sum\limits_{j\in S,\ j\neq i} \underline{p}_j}\right].$

15. If We Take Constraints Into Account, Not All Such Values Are Possible

- Reminder: $[\underline{q}_1, \overline{q}_1] = \left[\dfrac{\underline{p}_i}{\underline{p}_i + \sum\limits_{j\in S,\ j\ne i} \overline{p}_j}, \dfrac{\overline{p}_i}{\overline{p}_i + \sum\limits_{j\in S,\ j\ne i} \underline{p}_j}\right]$.
- Let us take $n = 10$, $\left[\underline{p}_i, \overline{p}_i\right] = [0, 0.2]$ for all i, and $S = \{1, \ldots, 9\}$.
- One can easily check that these intervals are already narrowed.
- The above formula leads to the upper bound

$$\overline{q}_1 = \frac{\overline{p}_1}{\overline{p}_1 + \sum\limits_{j\ne 1} \underline{p}_j} = \frac{0.2}{0.2 + 0 + \ldots + 0} = 1.$$

- However, $\overline{q}_1 = 1$ is only attained when $p_1 = 0.2$ and $p_2 = \ldots = p_9 = 0$, when $\sum\limits_{i=1}^{10} p_i = \sum\limits_{i=1}^{9} p_i + p_{10} \le 0.4 < 1$.

16. Conditional Probability: Bounds That Take Constraints Into Account

- *We know* the intervals $[\underline{p}_i, \overline{p}_i] \subseteq [0,1]$ and the condition $S \subset \{1, \ldots, n\}$.
- *We want to find*, for each $i \in S$, the range $[\underline{q}_i, \overline{q}_i]$ of values $q_i = \dfrac{p_i}{\sum\limits_{j \in S} p_j}$ when $p_j \in [\underline{p}_j, \overline{p}_j]$ and $\sum\limits_{j=1}^{n} p_j = 1$.
- *Solution:*

$$\underline{q}_i = \frac{\underline{p}_i}{\underline{p}_i + \min\left(\sum\limits_{j \in S,\ j \neq i} \overline{p}_j, 1 - \sum\limits_{k \notin S} \underline{p}_k - \underline{p}_i\right)};$$

$$\overline{q}_i = \frac{\overline{p}_i}{\overline{p}_i + \max\left(\sum\limits_{j \in S,\ j \neq i} \underline{p}_j, 1 - \sum\limits_{k \notin S} \overline{p}_k - \overline{p}_i\right)}.$$

17. Computing Conditional Probabilities: Asymptotically Optimal Algorithm

- Straightforward computations would require quadratic time: linear time for each of $2n$ values $\underline{q}_i$ and $\overline{q}_i$.
- However, we can compute these value in asymptotically optimal linear time if we first compute

$$\overline{P}_S \stackrel{\text{def}}{=} \sum_{j\in S} \overline{p}_j,\ \underline{P}_S \stackrel{\text{def}}{=} \sum_{j\in S} \underline{p}_j,\ \overline{P}_{-S} \stackrel{\text{def}}{=} \sum_{k\notin S} \overline{p}_k,\ \underline{P}_{-S} \stackrel{\text{def}}{=} \sum_{k\notin S} \underline{p}_k :$$

$$\underline{q}_i = \frac{\underline{p}_i}{\underline{p}_i + \min\left(\overline{P}_S - \overline{p}_i, 1 - \underline{P}_{-S} - \underline{p}_i\right)};$$

$$\overline{q}_i = \frac{\overline{p}_i}{\overline{p}_i + \max\left(\underline{P}_S - \underline{p}_i, 1 - \overline{P}_{-S} - \overline{p}_i\right)}.$$

18. Not All Combinations of q_i Are Possible

- Let's take $n = 8$, $\left[\underline{p}_i, \overline{p}_i\right] = [0.1, 0.15]$, and $S = \{1, 2, 3, 4\}$.
- For $p_1 = 0.15$, $p_2 = p_3 = p_4 = 0.1$, $p_5 = p_6 = p_7 = 0.15$, and $p_8 = 0.1$, we get $q_1 = \dfrac{0.15}{0.15 + 0.1 + 0.1 + 0.1} = \dfrac{1}{3}$.
- For $p_1 = 0.1$, $p_2 = p_3 = p_4 = 0.15$, $p_5 = p_6 = p_7 = 0.1$, $p_8 = 0.15$, we get $q_1 = \dfrac{0.1}{0.1 + 0.15 + 0.15 + 0.15} = \dfrac{2}{11}$.
- So, we can have $q_1 = \dfrac{1}{3}$, $q_2 = \dfrac{1}{3}$, $q_3 = \dfrac{2}{11}$, and $q_4 = \dfrac{2}{11}$.
- However, there are no p_i for which $q_i = \dfrac{p_i}{p_1 + p_2 + p_3 + p_4}$.
- Indeed, due to monotonicity, the only way to have $q_1 = \dfrac{1}{3}$ is to have $p_1 = 0.15$ and $p_2 = p_3 = p_4 = 0.1$.
- However, $q_2 = \dfrac{1}{3}$ is only possible for $p_1 = 0.1 \neq 0.15$.

19. Quantitative Conditional Possibility (Interval Case): Formulation of the Problem

- We know n intervals $\left[\underline{\mu}_i, \overline{\mu}_i\right]$.
- We assume that these intervals have already been narrowed.
- We know the condition S, i.e., a set $S \subset \{1, \ldots, n\}$.
- We want to find, for each $i \in S$,
  - the range $\left[\underline{q}_i, \overline{q}_i\right]$ of possible values of the quantitative conditional possibility

$$q_i = \frac{\mu_i}{\max\limits_{j \in S} \mu_j}$$

  - when $\mu_j \in \left[\underline{\mu}_j, \overline{\mu}_j\right]$ and $\max\limits_{1 \le j \le n} \mu_j = 1$.

20. Quantitative Conditional Possibility (Interval Case): Solution

- We want to find the range of $q_i = \dfrac{\mu_i}{\max\limits_{j \in S} \mu_j}$ when $\mu_j \in \left[\underline{\mu}_j, \overline{\mu}_j\right]$ and $\max\limits_{1 \le j \le n} \mu_j = 1$.
- If the set S contains all the indices i for which $\overline{\mu}_i = 1$, then $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$ for all $i \in S$.
- In all other cases:

$$\underline{q}_i = \frac{\underline{\mu}_i}{\max\left(\underline{\mu}_i, \max\limits_{j \in S,\ j \ne i} \overline{\mu}_j\right)}; \quad \overline{q}_i = \frac{\overline{\mu}_i}{\max\left(\overline{\mu}_i, \max\limits_{j \in S,\ j \ne i} \underline{\mu}_j\right)}.$$

- For each i, these formulas require n steps.
- So, a direct use of these formulas leads to a quadratic-time algorithm.

21. Asymptotically Optimal Computing of $\underline{q}_i$ and $\overline{q}_i$

- If the set S contains all indices i for which $\overline{\mu}_i = 1$, then we return the values $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$ for all $i \in S$.
- Otherwise, we compute the largest $\overline{M}$ and the second largest $\overline{S}$ of the values $\overline{\mu}_j$ corresponding to $j \in S$.
- We also compute the largest $\underline{M}$ and the second largest $\underline{S}$ of the values $\underline{\mu}_j$ corresponding to $j \in S$.
- If $\overline{\mu}_i = \overline{M}$, then $\underline{q}_i = \dfrac{\underline{\mu}_i}{\max\left(\underline{\mu}_i, \overline{S}\right)}$, else $\underline{q}_i = \dfrac{\underline{\mu}_i}{\max\left(\underline{\mu}_i, \overline{M}\right)}$.
- If $\underline{\mu}_i = \underline{M}$, then $\overline{q}_i = \dfrac{\overline{\mu}_i}{\max\left(\overline{\mu}_i, \underline{S}\right)}$, else $\overline{q}_i = \dfrac{\overline{\mu}_i}{\max\left(\overline{\mu}_i, \underline{M}\right)}$.
- We need to handle at least n input intervals.
- This algorithm is linear time $O(n)$ and is, thus, asymptotically optimal.

22. Not All Combinations of q_i Are Possible

- Let's take $n = 4$, $S = \{1,2,3\}$, $\left[\underline{\mu}_1, \overline{\mu}_1\right] = [0.1, 0.2]$, $\left[\underline{\mu}_2, \overline{\mu}_2\right] = \left[\underline{\mu}_3, \overline{\mu}_3\right] = [0.1, 0.5]$, $\left[\underline{\mu}_4, \overline{\mu}_4\right] = [1, 1]$.
- Then, $\left[\underline{q}_1, \overline{q}_1\right] = \left[\underline{q}_2, \overline{q}_2\right] = \left[\underline{q}_3, \overline{q}_3\right] = [0.2, 1]$.
- Let us show that it is impossible to have $q_1 = 0.5$, $q_2 = 0.2$, and $q_3 = 1.0$.
- Indeed, since $\mu_2 \geq 0.1$ and $\max(\mu_1, \mu_2, \mu_3) \leq 0.5$, we have $q_2 = \dfrac{\mu_2}{\max(\mu_1, \mu_2, \mu_3)} \leq \dfrac{0.1}{0.5} = 0.2$.
- The only possibility to have $q_2 = 0.2$ is when $\mu_2 = 0.1$ and $\max(\mu_1, \mu_2, \mu_3) = 0.5$.
- In this case, since $\mu_1 \leq 0.2$, we have $q_1 = \dfrac{\mu_1}{\max(\mu_1, \mu_2, \mu_3)} \leq \dfrac{0.2}{0.5} = 0.4$, so $q_1 \neq 0.5$.

23. Qualitative Conditional Possibility (Interval Case): Formulation of the Problem

- We know n intervals $\left[\underline{\mu}_i, \overline{\mu}_i\right]$.
- We assume that these intervals have already been narrowed.
- We know the condition S, i.e., a set $S \subset \{1, \ldots, n\}$;
- We want to find, for each $i \in S$,
  - the smallest $\underline{q}_i$ and the largest $\overline{q}_i$ of possible values of the qualitative conditional possibility
  - when $\mu_j \in \left[\underline{\mu}_j, \overline{\mu}_j\right]$ and $\max\limits_{1 \le j \le n} \mu_j = 1$.

24. Qualitative Conditional Possibility (Interval Case): Solution

- If the set S contains all the intervals for which $\overline{\mu}_i = 1$, then $\underline{q}_j = \underline{\mu}_j$ and $\overline{q}_j = \overline{\mu}_j$ for all j.
- Else, if $\underline{\mu}_i \geq \max\limits_{j \in S,\ j \neq i} \overline{\mu}_j$, then $\left[\underline{q}_i, \overline{q}_i\right] = [1, 1]$.
- Else, if $\overline{\mu}_i \geq \max\limits_{j \in S} \underline{\mu}_j$, then $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = 1$.
- Otherwise, $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$.
- For each i, these formulas require n steps.
- So, a direct use of these formulas leads to a quadratic-time algorithm.

25. Asymptotically Optimal Computing of $\underline{q}_i$ and $\overline{q}_i$

- If the set S contains all indices i for which $\overline{\mu}_i = 1$, then we return the values $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$ for all $i \in S$.
- Otherwise, we compute the maximum $\underline{M}$ of all the values $\underline{\mu}_i$, $i \in S$.
- Then, we compute the largest $\overline{M}$ and the second largest $\overline{S}$ of the values $\overline{\mu}_i$ corresponding to $i \in S$.
- After that, for each $i \in S$, we do the following:
  - if $\overline{\mu}_i = \overline{M}$ and $\underline{\mu}_i \geq \overline{S}$, we return $\underline{q}_i = \overline{q}_i = 1$;
  - if $\overline{\mu}_i < \overline{M}$ and $\underline{\mu}_i \geq \overline{M}$, we return $\underline{q}_i = \overline{q}_i = 1$;
  - otherwise, if $\overline{\mu}_i \geq \underline{M}$, we return $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = 1$;
  - for all other $i \in S$, we return $\underline{q}_i = \underline{\mu}_i$ and $\overline{q}_i = \overline{\mu}_i$.
- This algorithm takes linear time $O(n)$, and is, therefore, asymptotically optimal.

26. Not All Intermediate Values Are Possible

- For quantitative possibilities, all values q_i between $\underline{q}_i$ and $\overline{q}_i$ are possible.
- In contrast, for qualitative possibilities, many intermediate values are not possible.
- Example: $n = 3$, $S = \{1, 2\}$, $\left[\underline{\mu}_1, \overline{\mu}_1\right] = \left[\underline{\mu}_2, \overline{\mu}_2\right] = [0, 0.5]$ and $\left[\underline{\mu}_3, \overline{\mu}_3\right] = [1, 1]$.
- In this case, we have $\underline{q}_1 = 0$ and $\overline{q}_1 = 1$.
- However, it is not possible to have $q_i = 0.6 \in [0, 1]$.
- Indeed, each value q_i coincides either with 1, or with one of the original values μ_i.

27. Mean under Interval Uncertainty

- We want to find the range of $E = \sum\limits_{i=1}^{n} p_i \cdot x_i$ when $p_i \in [\underline{p}_i, \overline{p}_i]$ and $\sum\limits_{i=1}^{n} p_i = 1$.
- The max is attained when larger values have larger probability.
- So, if we sort x_i in increasing order $x_1 \le x_2 \le \ldots \le x_n$, we get $\overline{E} = \sum\limits_{i=1}^{k-1} \underline{p}_i \cdot x_i + p_k \cdot x_k + \sum\limits_{i=k+1}^{n} \overline{p}_i \cdot x_i$.
- Here, $p_k = 1 - \sum\limits_{i=1}^{k-1} \underline{p}_i - \sum\limits_{i=k+1}^{n} \overline{p}_i$, so $\underline{p}_k \le p_k \le \overline{p}_k$ implies that $\sum\limits_{i=1}^{k} \underline{p}_i - \sum\limits_{i=k+1}^{n} \overline{p}_i \le 1 \le \sum\limits_{i=1}^{k-1} \underline{p}_i - \sum\limits_{i=k}^{n} \overline{p}_i$.
- This enables us to find k in linear time. Since sorting is $O(n \cdot \log(n))$, we get total time $O(n \cdot \log(n))$.

28. Acknowledgments

- This work was supported in part:
  - by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence),
  - by the National Science Foundation grant DUE-0926721,
  - by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and
  - by grant N62909-12-1-7039 from the Office of Naval Research.
- This work was partly done when Salem Benferhat was visiting with the University of Texas at El Paso.