For Discrete-Time Linear Dynamical Systems under Interval Uncertainty, Predicting Two Moments Ahead Is NP-Hard

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1. Discrete-time dynamical systems are ubiquitous

- Most of the information about a physical system comes from measurements.
- From this viewpoint, knowing the state of the system means knowing the values of all the measured quantities.
- It is therefore reasonable:
  - to describe the state x of the system by the values of all these quantities, i.e.,
  - to associate the state with the tuple $x = (x_1, \ldots, x_n)$ of all these values.
- Many real-life systems are largely deterministic: their future state is almost uniquely determined by its current state.
- In mathematical terms, such systems are described by *discrete-time dynamical systems*.

2. Discrete-time dynamical systems are ubiquitous (cont-d)

- The state $x(t+1)$ at the next moment of time is uniquely determined by the current state $x(t)$:

$$x(t+1) = f(x(t)) \text{ for some function } f(x).$$

- For such systems, the state $x(t+1)$ uniquely determines the state $x(t+2)$, etc.
- Thus, all future states $x(t+k)$, for any $k > 0$, are uniquely determined by the current state $x(t)$.

3. Linear discrete-time dynamical systems are ubiquitous

- In many cases, the state changes are relatively small.
- So for a reasonable period of time, all the states $x(t)$ are close to the initial state $x(0)$.
- This means that the difference $\Delta x(t) \stackrel{\text{def}}{=} x(t) - x(0)$ is much smaller than $x(0)$.
- In this case, we have $x(t) = x(0) + \Delta x(t)$ and thus,

$$x(t+1) = f(x(t)) = f(x(0) + \Delta x(t)).$$

- Since $\Delta x(t) \ll x(0)$, terms which are quadratic (or higher order) in terms of $\Delta x(t)$ are much smaller than terms proportional to $\Delta x(t)$.
- For example, if $\Delta x(t) \approx 10\%$, then $(\Delta x(t))^2 \approx 1\%$, which is much smaller than 10%.

4. Linear discrete-time dynamical systems are ubiquitous (cont-d)

- In such cases, we can use an idea commonly used in physics:
  - we can expand the expression $f(x(t)) = f(x(0) + \Delta x(t))$ in Taylor series in terms of $\Delta x(t)$, and
  - ignore quadratic and higher order terms in this expansion.
- As a result, we get a linear expression describing the state $x(t+1)$ in terms of the state $x(t)$:

$$x_i(t+1) = a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j(t).$$

- Such dynamical systems are known as *linear*.

5. Need to take uncertainty into account

- In the ideal case:
  - when we know the initial state $x(t_0) = (x_1(t_0), \ldots, x_n(t_0))$ and we know all the parameters a_i and a_{ij},
  - then, by applying the above formula necessary amount of times, we can uniquely determine all future states.
- In practice, however, we rarely know both:
  - the exact values of the quantities that form the initial state and
  - the exact value of the dynamics-determining parameters.
- Indeed, most of our information comes from measurements and observations.
- Measurement and observations are never absolutely accurate.

6. Need to take uncertainty into account (cont-d)

- For each measured quantity q, the measurement result $\widetilde{q}$ is, in general, somewhat different from the actual (unknown) value q of this quantity.
- In other words, the difference $\Delta q \stackrel{\text{def}}{=} \widetilde{q} - q$, known as *measurement error*, is, in general, different from 0.
- It is desirable to take this uncertainty into account when predicting the future state of the system.

7. Case of interval uncertainty

- In many practical situations, the only information that we have about the measurement error Δq is the upper bound Δ on its absolute value:

$$|\Delta q| \le \Delta.$$

- In this case:
  - once we know the measurement result $\widetilde{q}$,
  - the only thing we can conclude about the actual value q of the corresponding quantity is that this value must belong to the interval $[\widetilde{q} - \Delta, \widetilde{q} + \Delta]$.
- This situation is known as *interval uncertainty*.
- Following tradition, we will denote interval-values quantities by boldface font.
- So x will usually denote and $\mathbf{x}$ will denote an interval.

8. Epistemic and aleatory uncertainty

- In order to formulate our problem in precise terms, it is important to distinguish between two different types of uncertainty.
- In some cases, there are some (unknown) actual values of the parameters a_i and a_{ij} that describe the dynamics at all moments of time.
- Our problem is that we do not know these values, all we know are intervals $[\underline{a}_i, \overline{a}_i]$ and $[\underline{a}_{ij}, \overline{a}_{ij}]$ that contains these parameters.
- This case is known as the case of *epistemic uncertainty*.
- In other cases, the values a_i and a_{ij} can actually change with time.
- The system may be in somewhat different environments, and this may affect how the state changes.
- For example, the dynamics of an underwater robot depends on the water density and current which, in general:
  - changes from location to location and
  - thus, changes with time.

9. Epistemic and aleatory uncertainty (cont-d)

- So, we have a more general expression:

$$x_i(t+1) = a_i(t) + \sum_{j=1}^{n} a_{ij}(t) \cdot x_j(t).$$

- In this case, at all moments of time, the corresponding values $a_i(t)$ and $a_{ij}(t)$ belong to the given intervals $[\underline{a}_i, \overline{a}_i]$ and $[\underline{a}_{ij}, \overline{a}_{ij}]$.
- This case, when the parameters can change within given intervals is known as the case of *aleatory uncertainty*.

10. Epistemic and aleatory uncertainty: example

- At first glance, the difference between epistemic and aleatory uncertainty may seem to be more philosophical than practical.
- However, the following simple example shows that these two types of uncertainty can lead to different results.
- Let's take $n = 1$, $x_1(0) = 1$, $a_1 = 0$ and $[\underline{a}_{11}, \overline{a}_{11}] = [-1, 1]$.
- In the case of aleatory uncertainty, the range of $x_1(1)$ is equal to $[-1, 1]$, and the range of $x_1(2)$ is equal to

$$[\underline{a}_{11}, \overline{a}_{11}] \cdot [\underline{x}_1(1), \overline{x}_1(1)] = [-1, 1] \cdot [-1, 1] = [-1, 1].$$

- In contrast, in the case of epistemic uncertainty, we have $x_1(1) = a_{11} \cdot x_1(0)$ and $x_1(2) = a_{11} \cdot x_1(1) = a_{11}^2 \cdot x_1(0)$. Here, $a_{11} \in [-1, 1]$.
- So the range of possible values of a_{11}^2 is $[0, 1]$.
- Since $x_1(0) = 1$, the range of $x_1(2)$ is equal to $[0, 1] \cdot 1 = [0, 1]$, which is different from the aleatory range $[-1, 1]$.

11. Prediction problem: aleatory case

- We know that the values $x_i(t)$ satisfy the equation

$$x_i(t+1) = a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j(t).$$

- We know the intervals $[\underline{x}_i(0), \overline{x}_1(0)]$ that contain $x_i(0)$.
- We know the intervals $[\underline{a}_i, \overline{a}_i]$ and $[\underline{a}_{ij}, \overline{a}_{ij}]$ that contain a_i and a_{ij}.
- We want to compute the range of all possible values of $x_i(t)$ for given i and t.

12. Prediction problem: epistemic case

- We know that the values $x_i(t)$ satisfy the equation

$$x_i(t+1) = a_i(t) + \sum_{j=1}^{n} a_{ij}(t) \cdot x_j(t).$$

- We know the intervals $[\underline{x}_i(0), \overline{x}_1(0)]$ that contain $x_i(0)$.
- We know the intervals $[\underline{a}_i, \overline{a}_i]$ and $[\underline{a}_{ij}, \overline{a}_{ij}]$ that contain the values $a_i(t)$ and $a_{ij}(t)$.
- We want to compute the range of all possible values of $x_i(t)$ for given i and t.

13. Main results

- For epistemic uncertainty, the problem of predicting one moment ahead is feasible.
- For aleatory uncertainty, the problem of predicting one moment ahead is feasible.
- For epistemic uncertainty, the problem of predicting two moments ahead is NP-hard.
- For aleatory uncertainty, the problem of predicting two moments ahead is NP-hard.

14. Auxiliary results

- When the initial state is known exactly, for aleatory uncertainty, the problem of predicting two moments ahead is feasible.
- For the case of epistemic uncertainty, it is not clear whether the problem of predicting two moments ahead is feasible.
- When the initial state is known exactly, for epistemic uncertainty, the problem of predicting three moments ahead is NP-hard.
- When the initial state is known exactly, for aleatory uncertainty, the problem of predicting three moments ahead is NP-hard.
- When the dynamics is known exactly, for epistemic uncertainty, the problem of predicting any number of moments ahead is feasible.
- When the dynamics is known exactly, for aleatory uncertainty, the problem of predicting any number of moments ahead is feasible.

15. Conclusions

- We showed that for linear discrete dynamical systems under interval uncertainty, predicting two moments ahead is NP-hard.
- It does not matter whether the uncertainty is epistemic or aleatoric, the problem is NP-hard in both cases.
- If we know the initial state exactly and the only uncertainty is in the coefficients of the dynamical system, then:
  - predicting two moments ahead is feasible, but
  - predicting three moments ahead is NP-hard.
- If we have the exact knowledge of the dynamical system, and the only uncertainty is in the initial state, then:
  - the prediction problem is feasible
  - no matter how many moments of time ahead we want to predict.

16. Remaining open problems

- The first remaining open problem is to find out whether for epistemic uncertainty, the problem of predicting two moments ahead is still feasible.
- The second open problem is related to continuous linear dynamical systems, i.e., systems described by linear differential equations

$$\dot{x}_i(t) = a_i + \sum_{j=1}^{n} a_{ij} \cdot x_j(t).$$

- Since a continuous dynamical system approximates discrete ones, our hypothesis is that:
  - for such systems under interval uncertainty,
  - prediction problem is NP-hard even when we have the exact knowledge of the initial state.
- It should be NP-hard in both cases: when the uncertainty is epistemic and when it is aleatory.

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18. Two general results that we will use in our proofs

- The first result is related to the fact that in some arithmetic expressions, each variable occurs only once.
- Such expressions are known as Single-Use Expressions (SUE, for short).
- It is known that for such expressions, we can compute the exact range if we simply:
  - replace each arithmetic operation
  - by the corresponding operation of interval arithmetic.
- This is know as *straightforward interval computations.*
- These computations require feasible – even linear – computation time.
- So, for these expressions, the problem of computing the exact range is feasible.

19. Two general results that we will use in our proofs (cont-d)

- The second result allows us to reduce:
  - the general dynamical system, with $a_i \neq 0$,
  - to the case when all the values a_i are zeros, i.e., when the dynamics equations take the following simplified form:

$$x_i(t+1) = \sum_{j=1}^{n'} a_{ij} \cdot x_i(t) \text{ or } x_i(t+1) = \sum_{j=1}^{n'} a_{ij}(t) \cdot x_i(t).$$

- Indeed, we can take $n' = n + 1$, and supplement the original values a_{ij} with the following new values:

$$a_{n'i} = 0 \text{ and } a_{in'} = a_i \text{ for all } i \leq n, \text{ and } a_{n'n'} = 1.$$

20. Predicting one moment ahead is feasible: proof

- Each desired value $x_i(1)$ is determined by the following formula:

$$x_i(1) = a_1 + a_{i1} \cdot x_1(0) + \ldots + a_{ij} \cdot x_i(0) + \ldots + a_{in} \cdot x_n(0).$$

- In this arithmetic expression, each of the inputs x_i, a_{ij} and $x_j(0)$ occurs only once.
- So, these are SUE expressions.
- Thus, these two problems are indeed feasible.

21. Predicting 2 moments ahead is NP-hard: proof

- It is known that the following problem is NP-hard:
  - given a square matrix $B = (b_{ij})$,
  - compute the range of the sum $\sum_{i=1}^{n} \sum_{j=1}^{n} s_i \cdot b_{ij} \cdot t_j$ when $\mathbf{s}_i = \mathbf{t}_j = [-1, 1]$ for all i and j.
- Now, for each $n \times n$ matrix B, we will consider the following $(2n+2) \times (2n+2)$ interval matrix $\mathbf{A} = \begin{pmatrix} 0 & \mathbf{U} \\ L & 0 \end{pmatrix}$, where:

$$L = \left(\begin{array}{c|ccc} 0 & \ldots & 0 & \ldots \\ \hline \ldots & & & \\ 0 & & B & \\ \ldots & & & \end{array}\right) \text{ and } \mathbf{U} = \left(\begin{array}{c|ccc} 0 & \mathbf{s}_1 & \ldots & \mathbf{s}_n \\ \hline \ldots & & & \\ 0 & & 0 & \\ \ldots & & & \end{array}\right).$$

- Please note that L is a traditional (number-valued) matrix, i.e., a degenerate case of an interval matrix.

22. Predicting 2 moments ahead is NP-hard: proof (cont-d)

- On the other hand, $\mathbf{U}$ has non-degenerate interval entries and is, thus, a truly interval matrix.
- Let us first consider the case of epistemic uncertainty, when the matrix A remains the same for all moments of time.
- In this case, for every matrix

$$A = \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} \in \mathbf{A} = \begin{pmatrix} 0 & \mathbf{U} \\ L & 0 \end{pmatrix}, \text{ we have}$$

$$A^2 = \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} = \begin{pmatrix} UL & 0 \\ 0 & LU \end{pmatrix}.$$

- Here,

$$UL = \left(\begin{array}{c|c} 0 & s^T \\ \hline y & 0 \end{array}\right) \left(\begin{array}{c|c} 0 & 0^T \\ \hline 0 & B \end{array}\right) = \left(\begin{array}{c|c} 0 & s^T B \\ \hline 0 & 0 \end{array}\right).$$

23. Predicting 2 moments ahead is NP-hard: proof (cont-d)

- Hence, for $x(0) = (0, \ldots, 0, t_1, \ldots, t_n)$, we have

$$x(2) = A^2 x(0) = U L x(0) = \left(\begin{array}{c|c} 0 & x^T B \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c} 0 \\ \hline t \end{array}\right) = \left(\begin{array}{c} s^T B t \\ \hline 0 \end{array}\right).$$

- Since computing the range of $s^T B t$ is NP-hard, computing the range of $x(2) = \mathbf{A}^2 x(0)$ is also an NP-hard problem.
- So, in the case of epistemic uncertainty, the corresponding prediction problem is indeed NP-hard.
- In the case of aleatory uncertainty, for every two matrices

$$A(0) = \begin{pmatrix} 0 & U_0 \\ L & 0 \end{pmatrix} \in \mathbf{A} = \begin{pmatrix} 0 & \mathbf{U} \\ L & 0 \end{pmatrix}, A(1) = \begin{pmatrix} 0 & U_1 \\ L & 0 \end{pmatrix} \in \mathbf{A} = \begin{pmatrix} 0 & \mathbf{U} \\ L & 0 \end{pmatrix},$$

$$A(1)A(0) = \begin{pmatrix} 0 & U_1 \\ L & 0 \end{pmatrix} \begin{pmatrix} 0 & U_0 \\ L & 0 \end{pmatrix} = \begin{pmatrix} U_1 L & 0 \\ 0 & L U_0 \end{pmatrix}.$$

24. Predicting 2 moments ahead is NP-hard: proof (cont-d)

- Here,

$$U_1 L = \left(\begin{array}{c|c} 0 & s^T \\ \hline y & 0 \end{array}\right) \left(\begin{array}{c|c} 0 & 0^T \\ \hline 0 & B \end{array}\right) = \left(\begin{array}{c|c} 0 & s^T B \\ \hline 0 & 0 \end{array}\right).$$

- Hence, for $x(0) = (0, \ldots, 0, t_1, \ldots, t_n)$, we have

$$x(2) = A(1)A(0)x(0) = U_1 L x(0) = \left(\begin{array}{c|c} 0 & x^T B \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c} 0 \\ \hline t \end{array}\right) = \left(\begin{array}{c} s^T B t \\ \hline 0 \end{array}\right).$$

- Since computing the range of $s^T B t$ is NP-hard, computing the range of $x(2) = A(1)A(0)x(0)$ is also an NP-hard problem.
- So, in the case of aleatory uncertainty, the corresponding prediction problem is also NP-hard.

25. When the initial state is known exactly, for aleatory uncertainty, predicting 2 moments ahead is feasible: proof

- In this case, we have $x_i(2) = \sum\limits_{j=1}^{n'} a_{ij}(1) \cdot x_j(1)$, where:

$$x_j(1) = \sum_{k=1}^{n'} a_{jk}(0) \cdot x_j(0).$$

- Substituting the expression for $x_j(1)$ into the formula for $x_i(2)$, we get the following expression:

$$x_i(2) = \sum_{j=1}^{n'} a_{ij}(1) \cdot \left(\sum_{k=1}^{n'} a_{jk}(0) \cdot x_j(0) \right).$$

- One can check that this is a single-use expression.
- Thus, the corresponding prediction problem is indeed feasible.

26. When the initial state is known exactly, predicting 3 moments ahead is NP-hard: proof

- We mentioned that the following problem is NP-hard:
  - given a square matrix $B = (b_{ij})$,
  - compute the range of the product $s^T Bt$, where $\mathbf{s}_i = \mathbf{t}_j = [-1, 1]$ for all i and j.
- For each $n \times n$ matrix B, we will consider the following $(2n+2) \times (2n+2)$ interval matrix: $\mathbf{A} = \begin{pmatrix} 0 & \mathbf{U} \\ L & 0 \end{pmatrix}$, where:

$$L = \left(\begin{array}{c|ccc} 0 & \ldots & 0 & \ldots \\ \hline \ldots & & & \\ 0 & & B & \\ \ldots & & & \end{array}\right); \ \mathbf{U} = \left(\begin{array}{c|ccc} 0 & \mathbf{s}_1 & \ldots & \mathbf{s}_n \\ \hline \mathbf{t}_1 & & & \\ \ldots & & 0 & \\ \mathbf{t}_n & & & \end{array}\right)$$

27. When the initial state is known exactly, predicting 3 moments ahead is NP-hard: proof (cont-d)

- Let us first consider the case of epistemic uncertainty.
- For every matrix

$$A = \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} \in \mathbf{A} = \begin{pmatrix} 0 & \mathbf{U} \\ L & 0 \end{pmatrix}, \text{ we have}$$

$$A^2 = \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} = \begin{pmatrix} UL & 0 \\ 0 & LU \end{pmatrix}.$$

- Hence

$$A^3 = A^2 A = \begin{pmatrix} UL & 0 \\ 0 & LU \end{pmatrix} \begin{pmatrix} 0 & U \\ L & 0 \end{pmatrix} = \begin{pmatrix} 0 & ULU \\ LUL & 0 \end{pmatrix}.$$

- Here,

$$UL = \left(\begin{array}{c|c} 0 & s^T \\ \hline t & 0 \end{array}\right) \left(\begin{array}{c|c} 0 & 0^T \\ \hline 0 & B \end{array}\right) = \left(\begin{array}{c|c} 0 & s^T B \\ \hline 0 & 0 \end{array}\right).$$

28. When the initial state is known exactly, predicting 3 moments ahead is NP-hard: proof (cont-d)

- Hence

$$ULU = \left(\begin{array}{c|c} 0 & s^T B \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c|c} 0 & s^T \\ \hline t & 0 \end{array}\right) = \left(\begin{array}{c|c} s^T Bt & 0 \\ \hline 0 & 0 \end{array}\right).$$

- So, for $x(0) = (1, 0, \ldots, 0)$, we have $x(3) = A^3 x(0)$ thus $x_1(3) = s^T Bt$.
- Since computing the range of $s^T Bt$ is NP-hard, computing the range of $x_1(3)$ is also an NP-hard problem.
- So, for the case of epistemic uncertainty, the statement is proven.
- Let us now consider the case of aleatory uncertainty.

29. When the initial state is known exactly, predicting 3 moments ahead is NP-hard: proof (cont-d)

- In this case, for all $t = 0, 1, 2$, for each matrix

$$A(t) = \begin{pmatrix} 0 & U_t \\ L & 0 \end{pmatrix} \in \mathbf{A} = \begin{pmatrix} 0 & \mathbf{U} \\ L & 0 \end{pmatrix}, \text{ we have}$$

$$A(2)A(1) = \begin{pmatrix} 0 & U_2 \\ L & 0 \end{pmatrix} \begin{pmatrix} 0 & U_1 \\ L & 0 \end{pmatrix} = \begin{pmatrix} U_2 L & 0 \\ 0 & L U_1 \end{pmatrix}.$$

- Hence

$$A(2)A(1)A(0) = (A(2)A(1))A(0) = \begin{pmatrix} U_2 L & 0 \\ 0 & L U_1 \end{pmatrix} \begin{pmatrix} 0 & U_0 \\ L & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & U_2 L U_0 \\ L U_1 L & 0 \end{pmatrix}.$$

30. When the initial state is known exactly, predicting 3 moments ahead is NP-hard: proof (cont-d)

- Here,

$$U_2L=\left(\begin{array}{c|c} 0 & s^T \\ \hline t & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & 0^T \\ \hline 0 & B \end{array}\right)=\left(\begin{array}{c|c} 0 & s^TB \\ \hline 0 & 0 \end{array}\right).$$

  Hence

$$U_2LU_0=\left(\begin{array}{c|c} 0 & s^TB \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & s^T \\ \hline t & 0 \end{array}\right)=\left(\begin{array}{c|c} s^TBt & 0 \\ \hline 0 & 0 \end{array}\right).$$

- So, for $x(0)=(1,0,\ldots,0)$, we have $x(3)=A(2)A(1)x(0)$, thus $x_1(3)=s^TBt$.

31. When the initial state is known exactly, predicting 3 moments ahead is NP-hard: proof (cont-d)

- Since computing the range of $s^T B t$ is NP-hard, computing the range of $x_1(3)$ is also an NP-hard problem.
- So, for the case of aleatory uncertainty, the statement is also proven.

32. When the dynamics is known exactly, the problem of predicting any number of moments ahead is feasible: proof

- When the dynamic coefficients a_{ij} are known exactly, the formulas take the following matrix form $x(t+1) = Ax(t)$.
- Here $x(t) \stackrel{\text{def}}{=} (x_1(t), \ldots, x_{n'}(t))$ and A is the matrix formed by the coefficients a_{ij}.
- In this case, we can easily show, by induction, that $x(t) = A^t x(0)$.
- In other words, each unknown value $x_i(t)$ is described by the formula:

$$x_i(t) = c_{i0} \cdot x_0(0) + \ldots + c_{in} \cdot x_n(0).$$

- Here c_{ij} denote the components of the matrix A^t.
- With respect to the only inputs which are known with interval uncertainty – the values $x_i(t)$ – this is a single-use expression.
- Thus, the corresponding prediction problems are indeed feasible.