Growth Rates under Interval Uncertainty

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1. General Systems: Linear Approximation

- *State:* is described by the values of the parameters $x_1, \ldots, x_n$.

- *Dynamic for continuous time:* $\dot{x}_i = f_i(x_1, \ldots, x_n)$.

- *Dynamic for discrete time:* $x_i(t+1) = f_i(x_1(t), \ldots, x_n(t))$.

- *Linearization:* $f_i(x_1, \ldots, x_n)$ is smooth, so in a small region, $f_i(x_1, \ldots, x_n) \approx b_i + \sum\limits_{j=1}^{n} a_{ij} \cdot x_j$.

- *Further simplification:* an appropriate shift $x_i \rightarrow x_i - s_i$ leads to $f_i(x_1, \ldots, x_n) = \sum\limits_{j=1}^{n} a_{ij} \cdot x_j$.

- *Resulting dynamic:* $\dot{x}_i = \sum\limits_{j=1}^{n} a_{ij} \cdot x_j$;

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij} \cdot x_j(t).$$

General Systems: . . .

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Acknowledgments

2. The Notion of a Growth Rate

- *New variables:* coefficients y_i in the expansion of $x_i(t)$ in eigenvectors of $A = (a_{ij})$.

- *Generic case* (all eigenvalues different):

$$\dot{y}_i = \lambda_i \cdot y_i, \text{ or } y_i(t+1) = \lambda_i \cdot y_i(t).$$

- *General solution:* a linear combination of

$$y_i(t) = y_i(0) \cdot \exp(\lambda_i \cdot t) \text{ or } y_i(t) = y_i(0) \cdot \lambda_i^t.$$

- *Asymptotically:* $x_i(t) \sim \exp(\lambda \cdot t)$ or $x_i(t) \sim \lambda^t$, where λ – *growth rate* – is the largest of the eigenvalues.

- *Applications:*

    - population growth,
    - growth in animals and plants,
    - spread of an epidemic,
    - financial growth.

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3. Growth under Interval Uncertainty

- *Idealized case – exactly known coefficients:* find the eigenvalues and find the largest of them.

- *In practice:* a_{ij} are only known with uncertainty.

- *Case of interval uncertainty:* we know
  - approximate values $\widetilde{a}_{ij}$, and
  - upper bounds Δ_{ij} on the approx. error

$$|\widetilde{a}_{ij} - a_{ij}|.$$

- Hence $a_{ij} \in \mathbf{a}_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}] \stackrel{\text{def}}{=} [\widetilde{a}_{ij} - \Delta_{ij}, \widetilde{a}_{ij} + \Delta_{ij}]$.

- *Problem:* find the interval $[\underline{\lambda}, \overline{\lambda}]$ of possible values of λ when $a_{ij} \in \mathbf{a}_{ij}$.

- *Alternative:* at least find an interval $I \supseteq [\underline{\lambda}, \overline{\lambda}]$.

- *Comment:* upper bound $\overline{\lambda}$ is of special interest: it determines, e.g., how fast a disease can spread.

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4. What Is Known

- *Case of small uncertainty:* $\Delta a_{ij} \stackrel{\text{def}}{=} \widetilde{a}_{ij} - a_{ij} \ll \widetilde{a}_{ij}$.

- *Solution:* linearize the dependence $\lambda_k(a_{ij})$, and get a range for λ.

- *General case:* the problem is computationally intractable (NP-hard).

- *Important case: non-negative matrix $a_{ij} \geq 0$.*

- *Examples:*
    - population growth,
    - spread of disease,
    - financial situations.

- *What we propose:* a new efficient algorithm that computes $\overline{\lambda}$ for non-negative matrices.

5. New Algorithm

- *We use:* a known algorithm $\mathcal{A}$ that computing $\lambda(A)$ for exact matrices A.

- *Input:* interval-valued matrix $\mathbf{A} = \|\mathbf{a}_{ij}\| = \|[\underline{a}_{ij}, \overline{a}_{ij}]\|$.

- *Algorithm:*

  – First, we apply $\mathcal{A}$ to $\underline{A} = \|\underline{a}_{ij}\|$; the resulting value is returned as $\underline{\lambda}$.

  – Then, we apply $\mathcal{A}$ to $\overline{A} = \|\overline{a}_{ij}\|$; the resulting value is returned as $\overline{\lambda}$.

- *Comment:* this idea only works for non-negative matrices.

6. Justification of the New Algorithm

- *Notation:* $A \leq B$ means $a_{ij} \leq b_{ij}$ for all i, j.

- *Notation:* $x \geq 0$ means $x_i \geq 0$ for all i.

- *Known fact:* $\lambda(A) = \max\limits_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2}$, where $\|x\|_2 \overset{\text{def}}{=} \sqrt{\sum\limits_{i=1}^{n} x_i^2}$ is the length of x.

- *Perron-Frobenius Theorem:* for $A \geq 0$, one of the eigenvectors x corresponding to the largest $\lambda(A)$ is non-negative: $x \geq 0$.

- *Conclusion:* $\lambda(A) = \max\limits_{x \geq 0 \,\&\, x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2}$.

- If $0 \leq A \leq B$ and $x \geq 0$, then $0 \leq Ax \leq Bx$ hence $\|Ax\|_2 \leq \|Bx\|_2$, $\dfrac{\|Ax\|_2}{\|x\|_2} \leq \dfrac{\|Bx\|_2}{\|x\|_2}$, and $\lambda(A) \leq \lambda(B)$.

- *Interval matrix:* for all $A \in [\underline{A}, \overline{A}]$, we have $\underline{A} \leq A \leq \overline{A}$, hence $\lambda(\underline{A}) \leq \lambda(A) \leq \lambda(\overline{A})$.

7. Case of Fuzzy Uncertainty

- *Situation:* often, we have uncertain expert estimates.

- *Fuzzy set description:* for each x, describe the degree $\mu(x) \in [0, 1]$ to which this value is possible.

- Reducing fuzzy uncertainty to interval uncertainty:

  - for each degree of certainty α, values possible with uncertainty $\geq \alpha$ form an interval (α-cut)

  $$\mathbf{x}(\alpha) = \{x \mid \mu(x) \geq \alpha\};$$

  - if we know α-cuts for all α, we can reconstruct $\mu(x)$.

- *Computation under fuzzy uncertainty:* for each level α, we apply the interval algorithm to the α-cuts.

- *Case of growth rate:* for $\alpha = 0, 0.1, \ldots, 1$, we apply our algorithm to α-cuts $\mathbf{a}_{ij}(\alpha)$.

- *Result:* intervals $[\underline{\lambda}, \overline{\lambda}](\alpha)$ form the fuzzy set for λ.

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8. Acknowledgments

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