# Towards an Optimal Algorithm for Computing Fixed Points: Dynamical Systems Approach, with Applications to Transportation Engineering

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# 1. Many Practical Situations Eventually Reach Equilibrium; Examples

- In *economics*,
  - a situation changes;
  - prices start changing (often fluctuating);
  - eventually, prices reach an equilibrium between supply and demand.

  In *transportation*,
  - a new road is built;
  - some traffic moves to this road to avoid congestion on the other roads;
  - this move causes congestion on the new road;
  - as a result, some drivers to go back to their previous routes;
  - etc.

# 2. Specific Challenges of Transportation Applications

- *Intuitively:* a road expansion (e.g., the opening of a new road) should always improve the traffic conditions.
- *In reality:* a new road can actually worsen traffic congestion:
  - too many cars move to a new road;
  - as a result, the new road becomes even more congested than the old roads initially were;
  - so the traffic situation will actually decrease.
- *Conclusion:* a possible negative effect of a new road on congestion.
- This negative effect is known as a paradox of transportation science.
- Due to this paradox, we need a detailed analysis in the planning of the new road.

## 3. It Is Often Desirable to Predict the Corresponding Equilibrium

- For the purposes of the long-term planning, it is desirable to find the corresponding equilibrium.
- *Economic example:* how, in the long run, oil prices will change if we start exploring new oil fields in Alaska?
- *Transportation example:* to what extent the introduction of a new road will relieve the traffic congestion?
- *General objective:* solve the practically important problem of predicting the equilibrium.
- *First step:* describe the equilibrium prediction problem in precise terms.

# 4. Finding an Equilibrium as a Mathematical Problem

- *Non-equilibrium states:* economic example:
  - *situation:* oil price is too high;
  - *result:* profitable to explore difficult-to-extract oil fields;
  - *new result:* the supply of oil increases, and prices drop.
- *Non-equilibrium states:* transportation example:
  - *situation:* too many cars move to a new road;
  - *results:* the new road becomes congested;
  - *new result:* drivers abandon the new road.
- *General description:* given a current state $x$, we can determine the state $f(x)$ at the next moment of time.
- *Equilibrium:* a state that does not change $f(x) = x$ (i.e., a fixed point).

# 5. Fixed Points in Transportation Engineering

- *Traffic demand:* # of drivers $d_{ij}$ who need to go from zone $i$ to zone $j$ – *origin-to-destination* (O-D) matrix.
- *Capacity* of a road link – the number $c$ of cars per hour which can pass through this link.
- *Travel time along a link*: $t = t^f \cdot \left[1 + a \cdot \left(\frac{v}{c}\right)^\beta\right]$, where:
    - $t^f = L/s$ is a *free-flow* time ($s$ is the speed limit),
    - $a \approx 0.15$ and $\beta \approx 4$ are empirical constants.
- *Equilibrium:* when
    - the travel time along all used alternative routes is exactly the same, and
    - the travel times along other un-used routes is higher.
- *Algorithms:* there exist efficient algorithms for finding the equilibrium.

## 6. Fixed Points in Transportation Engineering: Dynamic Case

- *We need to know:* O-D values $d_{ij}(t)$ for all $t$.
- *Problem:* if we build a new road, the values $d_{ij}(t)$ may change.
- *Example:*
  - the driver needs to be at work at 8:00 am;
  - the travel time is 30 minutes;
  - a new freeway decreases the expected travel time to 15 minutes.
- *Original decision:* leave home at 7:30 am.
- *New decision:* leave home at 7:45 am.
- *Conclusion:* $d_{ij}(7{:}30)\downarrow$ while $d_{ij}(7{:}45)\uparrow$.
- *Important:* take this choice of departure time into account.

# 7. How Drivers Select Departure Time

- *Logit model:* the probability $P_i$ that a driver will choose the $i$-th time interval is proportional to $\exp(u_i)$:

$$P_i = \frac{\exp(u_i)}{\exp(u_1) + \ldots + \exp(u_n)}.$$

- *Describing preferences:* empirical utility formula

$$u_i = -0.1051 \cdot E(T) - 0.0931 \cdot E(SDE) - 0.1299 \cdot E(SDL) -$$

$$1.3466 \cdot P_L - 0.3463 \cdot \frac{S}{E(T)},$$

where $E(X)$ means expected value,

  - $T$ is the travel time $T$,
  - $SDE$ is the wait time when arriving early,
  - $SDL$ is the delay when arriving late,
  - $P_L$ is the probability of arriving late, and
  - $S$ is the variance of the travel time.

## 8. Resulting Algorithm

- *Start:* with the 1st approximation O-D matrices $d_{ij}(t)$.
- *Iteration:*
  - Based on the values $d_{ij}(t)$, we find travel times at different time of the day $t$.
  - Based on the travel times, we estimate utilities $u_i$ of different start times.
  - From $u_i$, we compute the probabilities $P_i$ of selecting different start times.
  - Multiplying these probabilities by the number of drives who need to get from $i$ to $j$, we get new O-D matrices $d_{ij}(t)$.
  - Repeat until converges.
- *General case:* $x_{k+1} = f(x_k)$ (Picard iterations).

# 9. Situations when Picard's Iterations do not Converge: Economics

- *Main idea:* panicky fluctuations prevent convergence.
- *Example: reminder*
  - the price of oil is high;
  - it becomes profitable for companies to explore new oil fields;
  - as a result, the supply of oil will drastically increase, and the price of oil will go down.
- *Comment:* this is all done in a unplanned way.
- *Result:* supply exceeds demand.
- *Fact:* such fluctuations have happened in economics in the past.
- *Historical fact:* fluctuations have led to deep economic crises.

# 10. Situations when Picard's Iterations do not Converge: Transportation

- *Toy example illustrating a problem:*
  - now: no congestion, all start at 7:30, work at 8 am;
  - $M_1$: full O-D matrix for 7:30 am, 0 for 7:15 am;
  - based on this $M_1$, we get huge delays;
  - $M_2$: everyone leaves for work early at 7:15 am;
  - at 7:30, roads are freer, so in $M_3$, all start at 7:30;
  - no convergence: $M_1 = M_3 = \ldots \neq M_2 = M_4 \ldots$

# 11. A More Realistic Approach

- *Above idea:* drivers make decisions based only on *previous* day traffic.
- *More accurate idea:* drivers make decisions based on the *average* traffic over a few past days.
- *Resulting process:*
  - start with the 1st approximation O-D matrices $M_1$;
  - for $i = 2, 3, \ldots$:
    - compute the average $E_i = \dfrac{M_1 + \ldots + M_i}{i}$,
    - find traffic times based on $E_i$;
    - use these traffic times to compute a new O-D matrix $M_{i+1} = F(E_i)$;
    - repeat until converges.
- *Process converges:* on toy examples, on El Paso network, etc.

## 12. Algorithm Simplified

- *Main idea:* once we know the previous average $E_i$, we can compute

$$E_{i+1} = \frac{(M_1 + \ldots + M_i) + M_{i+1}}{i+1} = \frac{i \cdot E_i + M_{i+1}}{i+1} =$$

$$E_i \cdot \left(1 - \frac{1}{i+1}\right) + M_{i+1} \cdot \frac{1}{i+1}.$$

- *We know:* that $M_{i+1} = F(E_i)$.
- *Resulting algorithm:*
  - start with the 1st approximation O-D matrices

$$E_1 = M_1;$$

  - compute $E_{i+1} = E_i \cdot \left(1 - \frac{1}{i+1}\right) + F(E_i) \cdot \frac{1}{i+1};$
  - repeat until converges.

# 13. Towards an Optimal Algorithm for Computing Fixed Points

- *Idea:* when iterations $x_{k+1} = f(x_k)$ do not converge,
$$x_{k+1} = x_k + \alpha \cdot (f(x_k) - x_k) = (1 - \alpha_k) \cdot x_k + \alpha_k \cdot f(x_k).$$
- *Question:* which choice of $\alpha_k$ is best?
- *Idea:* this is a discrete approximation to a continuous-time system $\dfrac{dx}{dt} = \alpha(t) \cdot (f(x) - x)$.
- *Scale invariance:* the system should not change if we use a different discretization, i.e., re-scale $t$ to $t' = t/\lambda$:
$$\frac{dx}{dt'} = (\lambda \cdot \alpha(\lambda \cdot t')) \cdot (f(x) - x).$$
- *Conclusion:* $\lambda \cdot \alpha(\lambda \cdot t') = a(t')$, so for $\lambda = 1/t'$, we get $\alpha(t') = \dfrac{c}{t'}$ for some $c$.
- *Fact:* this is exactly what we used: $\alpha_k = 1/k$.

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## 15. Logit Discrete Choice Model: A New Justification

- *Reasonable assumption:* if we add same incentive to all routes, probabilities will not change.
- *For 2 routes:* $P_1 = F(\Delta V)$, where $\Delta V \stackrel{\text{def}}{=} V_1 - V_2$.
- *Bayes theorem:*

$$P(H_i \mid E) = \frac{P(E \mid H_i) \cdot P_0(H_i)}{P(E \mid H_1) \cdot P_0(H_1) + \ldots + P(E \mid H_n) \cdot P_0(H_n)}.$$

- *Idea:* if we add an incentive $v_0$ to one of the routes, this changes the probability of selecting this route:

$$F(\Delta V + v_0) = \frac{A(v_0) \cdot F(\Delta V)}{A(v_0) \cdot F(\Delta V) + B(v_0) \cdot (1 - F(\Delta V))}.$$

- *Conclusion:* $F(\Delta V) = \dfrac{1}{1 + e^{-\beta \cdot \Delta V}}$, so

$$p_1 = F(V_1 - V_2) = \frac{e^{\beta \cdot V_1}}{e^{\beta \cdot V_1} + e^{\beta \cdot V_2}}.$$

# 16. Taking Uncertainty into Account

- *Deterministic model:* $t = t^f \cdot \left[1 + a \cdot \left(\frac{v}{c}\right)^\beta\right]$.
- *Traffic assignment:* a driver minimizes the travel time $t = t_1 + \ldots + t_n$.
- *In practice:* travel times vary.
- *Decision theory:* maximize expected utility $E[u]$.
- *How utility depends on travel time:* $u(t) = -U(t)$, where $U(t) = \exp(\alpha \cdot t)$.
- *Conclusion:* the driver minimizes

$$E[U(t)] = E[\exp(\alpha \cdot t)] = E[\exp(\alpha \cdot (t_1 + \ldots + t_n)] =$$

$$E[\exp(\alpha \cdot t_1) \cdot \ldots \cdot \exp(\alpha \cdot t_n)].$$

- Deviations on different links are independent, so

$$E[U(t)] = E[\exp(\alpha \cdot t_1)] \cdot \ldots \cdot E[\exp(\alpha \cdot t_n)].$$

## 17. Taking Uncertainty into Account (cont-d)

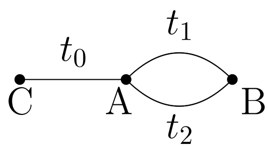
- Minimizing $E[U(t)] = E[\exp(\alpha \cdot t_1)] \cdot \ldots \cdot E[\exp(\alpha \cdot t_n)]$ $\Leftrightarrow$ minimizing $\sum\limits_{i=1}^{n} \widetilde{t}_i$, where $\widetilde{t}_i \stackrel{\text{def}}{=} \ln(E[\exp(\alpha \cdot t_i)])$.
- $\widetilde{t}$ depends on $t^f$ and $r \stackrel{\text{def}}{=} \dfrac{\overline{t} - t^f}{t}$: $\widetilde{t} = F(t^f, r)$.
- If we divide a link into sublinks, we conclude that $F(t_1^f + t_2^f, r) = F(t_1^f, r) + F(t_2^f, r)$, hence $\widetilde{t} = t^f \cdot k(r)$.
- For no-congestion case $r = 0$, we have $\widetilde{t} = t^f$, so $k(0) = 1$ and $k(r) = 1 + a_0 \cdot r + a_2 \cdot r_2 + \ldots$
- *Empirical analysis:* $a_1 \approx 1.4$, $b \approx 0$, so

$$\widetilde{t} = t^f \cdot \left[1 + a \cdot a_1 \cdot \left(\frac{v}{c}\right)^{\beta}\right].$$

- *Solution:* use the standard travel time formula with $a \cdot a_1 \approx 0.21$ instead of $a \approx 0.14$.

# 18. Exponential Disutility Functions in Transportation Modeling: Justification

- *Situation:*




- *Reasonable assumption:* the driver starting at C will choose the same road as the driver starting at A.

- *Formally:* if $E[u(t_1)] < E[u(t_2)]$ then
$$E[u(t_1 + t_0)] < E[u(t_2 + t_0)].$$

- *Result:* $u(t) = t$, $u(t) = \exp(c \cdot t)$, or
$$u(t) = -\exp(-c \cdot t).$$

- *Fact:* this is exactly the empirically justified formula used in transportation.



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