Fuzzy Ideas Explain Fechner Law and Help Detect Relation Between Objects in Video

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1. Practical problem with which we started this research

- In a video, we usually have several objects that appear from time to time.
- One of the things that we need to know in order to understand the video is:
  - which pairs of objects are related – and if they are related, to what extent, and
  - which pairs of objects are not related to all.

2. A seemingly natural idea

- If the objects are closely related – e.g., a computer and its mouse – then in most cases, they will appear or not appear at the same time.
- Yes, there will be some cases when we see the computer and not the mouse – and vice versa, the mouse but not the computer.
- However, in general, they would appear approximately the same number of times.
- On the other hand:
  - if two objects are not related,
  - then it is highly improbable that these two unrelated objects will appear the exact number of times.
- It therefore seems reasonable to take:
  - the difference between the numbers of appearances of the two objects
  - as a measure of the degree to which these objects are not related.

3. A direct implementation of this idea and why it does not work

- In a nutshell, the above idea means that:
  - the smaller the difference between the two numbers of occurrence,
  - the more probable it is that the two objects are related.
- Let us explain why there is a problem with the direct implementation of the above idea.
- For this purpose, let us consider the following two situations.
- In the first situation, we have two related objects that most frequently appear together – e.g., in 90% of the cases.
- In this situation, the numbers of occurrences can be, e.g., 90 and 100.
- The difference between these two numbers is $|100 - 90| = 10$, and the objects are related.

4. A direct implementation of this idea and why it does not work (cont-d)

- In the second situation, we have two unrelated objects that occur rarely in the video.
- For example, one of the two objects appeared 1 time, and another one appeared 5 times.
- In this case, the difference between the numbers of occurrences is $|5 - 1| = 4$, which is much smaller than 10.
- However, we can hardly make a conclusion that these two objects are related.
- In this example, we have small difference for unrelated objects and a larger difference for related ones – contrary to the above idea.

5. Similar examples from other application areas

- A similar problem appears in other application areas; for example:
  - if we look for economically similar folks,
  - at first glance, a seemingly reasonable idea is to use the difference between the yearly incomes as a measure of their dissimilarity.
- However:
  - the economic difference between a poor student who gets \$20K per year and a professor who gets \$70K per year is huge, while
  - the two billionaires whose annual incomes are \$2 billion and \$4 billion are economically similar.
- On the other, the difference $|70 - 20| = 50$K in the first situation is much smaller than the 2-billion difference in the second situation.

6. Similar examples from other application areas (cont-d)

- Another example: we can easily see the difference between a dim 20W light bulb and a reasonable 40W bulb.
- However:
  - we will probably not see that much different between 100W and 130W bulbs,
  - even though for the second pair, the difference is larger.

7. How can we modify this idea to make it work?

- A natural idea is to *re-scale* the numbers of occurrences before computing the difference, i.e., compute the difference:
  - not between the numbers of occurrences $|a - b|$,
  - but between the appropriate re-scaled numbers.
- In other words, we should consider the difference $|f(a)-f(b)|$ between the values $f(a)$ and $f(b)$.
- Here, $f(a)$ is an appropriate re-scaling function $f(a)$ from positive numbers to positive numbers.
- To utilize this idea, we need to select an appropriate function $f(a)$.

8. What we do in this talk

- The problem of selecting a re-scaling is not mathematically well-defined.
- It is formulated by using imprecise terms from natural language like "appropriate".
- So, to solve this problem, it makes sense to use ideas from fuzzy techniques.
- Indeed, fuzzy techniques were designed by Lotfi Zadeh specifically for translating imprecise ("fuzzy") knowledge into precise rules.
- In this talk, we show that these ideas indeed lead to a reasonable selection of the re-scaling function.
- The resulting selection indeed helps to detect relation between objects in video.
- As a side effect, it explains an empirical Fechner law about human perception.

9. We are interested in situations when the difference is relatively small

- Sometimes, the absolute value $|\Delta a|$ of the difference $\Delta a \stackrel{\text{def}}{=} b - a$ between the numbers of occurrences is huge.
- In such cases, clearly the corresponding objects are not related.
- The difficult-to-decide cases is when this difference is relatively small.
- So, in this talk, we will concentrate on these cases.

10. Let us use linearization

- When the difference is small, what can we say about the difference $f(b) - f(a) = f(a + \Delta a) - f(a)$ between the re-scaled values?
- In this case, we can use the idea actively used in physics:
  - expand this expression in Taylor series in terms of the small difference and
  - only keep the first non-zero term in this expansion.
- For our expression, this idea leads to

$$|f(a + \Delta a) - f(a)| \approx |(f(a) + f'(a) \cdot \Delta a) - f(a)| = F(a) \cdot |\Delta a|.$$

- Here, we denoted $F(a) \stackrel{\text{def}}{=} f'(a)$.

11. What are reasonable properties of the function $F(a)$

- The larger the value of a, the less important is the difference – e.g., for billionaires, there is practically no difference.
- On the other hand, when a is small, even a small difference is very important.
- For a little kid who does not have any money of his own, a dollar is a huge amount.
- However, for a usual adult, adding $1 would not make a big change in their happiness.
- Since fuzzy technique usually deals with rules, let us formulate the above argument in terms of rules.

12. What are reasonable properties of $F(a)$ (cont-d)

- The first bullet point can be described by the following rule:

  When the value a is large, the value $F(a)$ is small.

- Similarly, the second bullet point can be described by the following rule:

  When the value a is small, the value $F(a)$ is large.

13. These two rules are actually similar, and what can we conclude based on this similarity

- At first glance, the above two rules are rather different.
- However, a simple re-formulation makes them similar.
- Indeed, the intuition behind the second rule can also described in the following form:

When the value $F(a)$ is large, the value a is small.

- Now, this is very similar to the first rule, the only difference is that:
  - the first rule described the transformation from a to $F(a)$, while
  - the reformulated second rule describes the transition from $F(a)$ to a.
- We want to translate these informal rules to a precise expression.
- These two transition are described by the exact same rule.

14. These two rules are actually similar, and what can we conclude based on this similarity (cont-d)

- So, it makes sense to conclude that these two transitions are described by the same function.
- In the first rule, we apply the function $F(a)$ to the value a and get $F(a)$.
- Similarly, in the reformulated second rule, we should apply the same function F to the value $F(a)$ and get a.
- In other words, we should get $F(F(a)) = a$ for all a.
- Functions that satisfy this property are known as *involutions*.
- In these terms, we have the following question:

Which involution should we use?

15. We will show the two ways to answer this question

- In this talk, we will use two different ideas to answer this question.
- We will see that both ideas leads to the exact same conclusion.

16. First idea: simplicity

- Our first idea is to utilize one of Zadeh's principles – the principle of simplicity:
  - when we have several options,
  - we should always select the simplest one.
- Simplicity is not just a philosophical idea.
- It has been proven that:
  - for an appropriate formalization of simplicity,
  - this idea leads to asymptotically correct determination of the dependence from data.

17. How can we define simplicity

- Data processing is usually performed by computers; intuitively:
  - simple algorithms are fast to compute, while
  - complex algorithms require longer computation time.
- How can we gauge the computation time?
- In a computer, the only hardware supported operations are arithmetic operations: addition, substraction, multiplication, and division.
- Whatever the computer computes, it performs a sequence of arithmetic operations.
- For example:
  - when we ask the computer to compute e^x,
  - what is actually computes is the sum of the first few terms in the corresponding Taylor series:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}.$$

18. How can we define simplicity (cont-d)

- It is therefore reasonable to gauge the computation time:
  - by counting the number of additions, subtractions, multiplications, and divisions in the corresponding computation, and
  - by multiplying these numbers by the time needed to perform the corresponding arithmetic operation.
- We know the relation between these times:
  - addition and subtraction are the fastest operations,
  - multiplication is slower – since it implies several additions, and
  - division is the slowest – since it involves several multiplications.

19. So what is the simplest involution?

- Can we form the function $F(a)$ by using only faster arithmetic operations, i.e., addition, subtraction, and multiplication?
- Not really, since by using these operations, all we get is a polynomial of a, and all polynomials tend to infinity when a increases.
- However, we wanted a function for which for large a, the value $F(a)$ is small.
- In particular, for which $F(a)$ should not tend to infinity as a increases.
- So, to get the desired function, we need to use at least one division.
- The simplest case would be if we have exactly one division and no other arithmetic operations.
- We start with the variable a and constants c, and we need to involve a in our computations.

20. So what is the simplest involution?

- So, we have two choices: $F(a) = \dfrac{a}{c}$ and $F(a) = \dfrac{c}{a}$.
- The first choice does not work, since it also leads to $F(a) \to \infty$ when a increases.
- For the second choice, the condition that $F(F(a)) = a$ means $\dfrac{c}{c/a} = a$ and is, thus, automatically satisfied; so:
  - out of all possible involutions $F(a)$ for which large a lead to small $F(a)$,
  - the simplest are the functions of the type $F(a) = c/a$ for some constant c.

21. Second idea

- Instead of looking for the simplest involution, let us look for all possible involutions:
  - that can be exactly computed,
  - i.e., that can be computed by a finite sequence of arithmetic operations.
- One can easily check that:
  - if we start with the variable a and apply a finite number of arithmetic operations,
  - then what we get are all *rational functions*, i.e., all ratios of two polynomials

$$R(a) = \frac{c_0 + c_1 \cdot a + \ldots + c_n \cdot a^n}{c'_0 + c'_1 \cdot a + \ldots + c'_m \cdot a^m}.$$

22. Second idea (cont-d)

- Indeed, each such ratio can be computed if:
  - we use addition and multiplication to compute the values of the numerator and the denominator, and
  - then we apply one division to compute the ratio.
- Vice versa, one can easily show that:
  - the variable and the constants are rational function, and
  - the sum, difference, product, and ratio of two rational functions are also rational function.
- Thus, by induction, any sequence of arithmetic operations results in a rational function.
- So, the second idea simply means that we are looking for rational functions that are involutions.

23. Second idea (cont-d)

- Such functions can be described by the following proposition:
- *Let $F(a)$ be an involution from positive numbers to positive numbers for which $F(a)$ does not tends to infinity as a increases.*
- *Then, $F(a) = c/a$ for some $c > 0$.*
- So, we arrive at the same conclusion as when we used the first idea.

24. What is the resulting transformation

- Once we have the fuzzy-motivated condition that the function $F(a)$ is an involution, both our ideas lead to the same result: that $F(a) = c/a$.
- We may recall that $F(a)$ is the derivative of the desired transformation function $f(a)$: $F(a) = f'(a)$.
- Thus, we conclude that $f(a) = c \cdot \ln(a) + C$ for some constant C.
- Of course, for computing the differences $|f(a) - f(b)|$, the constant C is irrelevant: it disappears when we compute the difference $f(a)-f(b)$.
- Thus, without losing generality, we can state that the resulting transformation function has the form $f(a) = c \cdot \ln(a)$ for some c.

25. This helps detect relation between objects in a video, and explains the Fechner Law

- The use of logarithms also helps to solve the problem that we started with: detecting relation between objects in a video.
- Fechner Law about human perception states that:
  - the perceived difference between two stimuli a and b (be they audio, visual, or others)
  - is determined by the difference $|\ln(a) - \ln(b)|$ between the logarithms of their intensities.
- Thus, our fuzzy-motivated ideas explain the Fechner Law.

26. Conclusions and possible future work

- In this talk, we use fuzzy ideas to explain the efficiency of logarithmic re-scaling both:
  - in detecting relation between objects in video and
  - in describing human perception.
- In this result, we have applied simple fuzzy ideas to a reasonably simple relation-detecting algorithm based on video embedding.
- In the future, it is desirable to extend this result:
  - to other video-processing algorithms – e.g., ones based on deep learning, and
  - to more complex fuzzy ideas, e.g., interval-valued and, more generally, type-2 fuzzy systems (or maybe even type-3?).

27. Proof of the Proposition

- Let $F(a)$ be a rational function which is an involution, i.e., for which $F(F(a)) = a$ for all positive values a.
- Let us assume that $F(a)$ does not tend to infinity as a increases.
- Let us prove that we then have $F(a) = c/a$ for some constant c.
- Let us first prove that the function $F(a)$ is one-to-one, i.e., that:
  - for different positive numbers $a \neq b$,
  - the values $F(a)$ and $F(b)$ should be different.
- Indeed, if these values were equal, i.e., if we had $F(a) = F(b)$, then we will have $F(F(a)) = F(F(b))$.
- However, since $F(a)$ is an involution, that would imply $a = b$.
- But we assumed that $a \neq b$. This contradiction shows that we cannot have $F(a) = F(b)$.
- Thus, $a \neq b$ indeed implies that $F(a) \neq F(b)$.

28. Proof of the Proposition (cont-d)

- Let us now prove that the function $F(a)$ is an onto function, i.e., that every positive number a is equal to $F(x)$ for some positive number x.
- Indeed, this is true for $x = F(a)$.
- So, $F(a)$ is one-to-one onto function. Such functions are known as bijections.
- Every rational function is continuous everywhere it is defined, so $F(a)$ is a continuous function.
- The function $F(a)$ is a continuous bijection from the set of all positive numbers to itself.
- It is known that all such functions are either strictly increasing or strictly decreasing.
- Strictly increasing means that for $a < b$ we always have $F(a) < F(b)$.
- Strictly decreasing means that for $a < b$ we always have $F(a) > F(b)$.

29. Proof of the Proposition (cont-d)

- Let us prove this by contradiction.
- Since the function $F(a)$ is one-to-one, when $a < b$, we cannot have $F(a) = F(b)$, we must have either $F(a) < F(b)$ or $F(a) > F(b)$.
- We want to prove that if we have $F(a) > F(b)$ for some $a < b$, then we cannot have $F(a') > F(b')$ for some $a' < b'$.
- Indeed, for every $\alpha \in (0,1)$, we can conclude that $\alpha \cdot a < \alpha \cdot a'$ and $(1-a) \cdot b < (1-\alpha) \cdot b'$ and thus, $\alpha \cdot a + (1-\alpha) \cdot b < \alpha \cdot b + (1-\alpha) \cdot b'$.
- For $\alpha = 0$ and $\alpha = 1$, this inequality is also true – it corresponds to $a < b$ and $a' < b'$.
- Let us now consider the function that maps $\alpha \in [0,1]$ into the difference $F(\alpha \cdot a + (1-\alpha) \cdot a') - F(\alpha \cdot b + (1-\alpha) \cdot b')$.
- This function is a composition of continuous functions and is, therefore, continuous itself.

30. Proof of the Proposition (cont-d)

- When $\alpha = 0$, this difference is equal to $F(a') - F(b')$ and is, therefore, positive.
- When $\alpha = 1$, this difference is equal to $F(a) - F(b)$ and is, therefore, negative.
- Thus, by the Intermediate Value Theorem for continuous functions, there exists an $\alpha \in (0, 1)$ for which this difference is equal to 0, i.e.:
  - for which $\alpha \cdot a + (1 - \alpha) \cdot b < \alpha \cdot b + (1 - \alpha) \cdot b'$ but
  - for which
  $$F(\alpha \cdot a + (1 - \alpha) \cdot a') = F(\alpha \cdot b + (1 - \alpha) \cdot b').$$
- This contradicts to the fact that the function $F(a)$ is one-to-one.
- Thus, the function $F(a)$ is indeed either strictly increasing or strictly decreasing.
- We assumed that the function $F(a)$ is not strictly increasing, thus it must be strictly decreasing.

31. Proof of the Proposition (cont-d)

- Since this function is onto, values close to 0 must map to values close to infinity and vice versa, i.e., we must have $F(a) \to 0$ as $a \to \infty$.
- Let us now deal with the natural extension of a function $F(a)$ to all complex numbers.
- Every rational function is analytical, so our involution is analytical too.
- In particular, we can naturally extend it to all complex values z.
- Of course, for some z, the denominator can be 0, so for these z, we will have $F(z) = \infty$.
- The composition of rational functions is also rational, so the function $F(F(z))$ is also analytical.

32. Proof of the Proposition (cont-d)

- It is known that:
  - if two analytical functions are equal on an infinite set that contains a limit point,
  - then these two functions are equal for all z.
- In our case, the functions $F(F(z))$ and z are equal for all positive real numbers z.
- The set of all positive numbers clearly contains many limit points, so we conclude that $F(F(z)) = z$ for all complex values z.
- In complex domain:
  - each polynomial of each degree n – including the numerator and the denomination of the expression $F(z)$
  - has exactly n roots $z_1, \ldots, z_n$ – if we count roots with multiplicity the corresponding number of times.

33. Proof of the Proposition (cont-d)

- Thus, each polynomial can be represented as a product

$$c \cdot (z - z_1) \cdot \ldots \cdot (z - z_n).$$

- So, the rational function $F(z)$ can be represented as the ratio of two such products:

$$F(z) = \frac{c \cdot (z - z_1) \cdot \ldots \cdot (z - z_n)}{c' \cdot (z - z'_1) \cdot \ldots \cdot (z - z'_m)}.$$

- We can simplify this expression if we divide both numerator and denominator by c'.
- Then the constant c' in the denominator disappears; also:
  - if both the numerator and the denominator contain the same difference $z - z_i$,
  - then we can divide both numerator and denominator by this difference.

34. Proof of the Proposition (cont-d)

- Thus, we can conclude that the function $F(z)$ has the following form:
$$F(z) = \frac{c \cdot (z - z_1) \cdot \ldots \cdot (z - z_n)}{(z - z'_1) \cdot \ldots \cdot (z - z'_m)}.$$
- Here, all the values z_i are different from all the values z'_j.
- Let us show that all the roots in the numerator are equal to each other.
- We will prove this by contradiction. Suppose that we have $z_i \neq z_j$ for some $i \neq j$.
- Then we would have $F(z_i) = F(z_j) = 0$. Thus, $F(F(z_i)) = F(F(z_j))$.
- On the other hand, since $F(z)$ is an involution, we should have $F(F(z_i)) = z_i \neq F(F(z_j)) = z_j$.
- This contradicts to our conclusion that $F(F(z_i)) = F(F(z_j))$.
- This contradiction shows that the roots in the numerator cannot be different.

35. Proof of the Proposition (cont-d)

- Similarly, we can prove that all the roots in the denominator are equal to each other.
- We can also prove it by contradiction.
- Indeed, suppose that we have $z'_i \neq z'_j$ for some $i \neq j$.
- Then we would have $F(z'_i) = F(z'_j) = \infty$.
- Thus, $F(F(z'_i)) = F(F(z'_j))$.
- On the other hand, since $F(z)$ is an involution, we should have $F(F(z'_i)) = z'_i \neq F(F(z_j)) = z'_j$.
- The contradiction shows that these roots cannot be different.
- So, the function $F(z)$ has the form

$$F(z) = \frac{c \cdot (z - z_1)^n}{(z - z'_1)^m}, \text{ for some } n \geq 0 \text{ and } m \geq 0.$$

- Since we want $F(a) \to 0$ as $a \to \infty$, we must have $m > n$.

36. Proof of the Proposition (cont-d)

- Since $n \geq 0$, this means that $m \geq 1$, so there is some value z'_1.
- For $z = z'_1$, we have $F(z'_1) = \infty$.
- We know that $F(\infty) = 0$, thus $F(F(z'_1)) = 0$.
- Since the function $F(z)$ is an involution, we have $F(F(z'_1)) = z'_1$, thus $z'_1 = 0$.
- So, the function $F(z)$ has the form

$$F(z) = \frac{c \cdot (z - z_1)^n}{z^m}, \text{ for some } m > 0.$$

- In this case, $F(0) = \infty$.
- What happens if $n > 0$?
- In this case, we will have $F(z_1) = 0$.
- We know, however, that $F(0) = \infty$, so $F(F(z_1)) = F(0) = \infty$.
- This contradicts to the involution equality $F(F(z_1)) = z_1$.

37. Proof of the Proposition (cont-d)

- This contradiction shows that we cannot have $n > 0$, so $n = 0$.
- Thus, the function $F(z)$ has the form $F(z) = \dfrac{c}{z^m}$.
- For the above expression, the involution requirement takes the form

$$F(F(z)) = \frac{c}{(c/z^m)^m} = \frac{c}{c^m/z^{m^2}} = c^{1-m} \cdot z^{m^2} = z.$$

- This equality must be satisfied for all z, so we must have $m^2 = 1$.
- Thus – since m is a positive number – we must have $m = 1$.
- In this case, the desired involution equality is always satisfied,
- So, we indeed conclude that $F(z) = c/z$ for some $c > 0$.
- The proposition is proven.

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