Fuzzy Sets Can Be Interpreted as Limits of Crisp Sets, and This Can Help to Fuzzify Crisp Notions

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1. Fuzzy Sets: Reminder

- Many properties are well-defined and objective ("crisp").
- For example, a real number x is either positive or not positive, it is either smaller than 10 or not, etc.
- Each such crisp property P can be described by a (crisp) *set* S of all the objects that satisfy P.
- Humans routinely deal with properties which are not fully well-defined, such as "small", "young", etc.
- To deal with such imprecise ("fuzzy") properties, L. Zadeh introduced the notion of a *fuzzy set*.
- A *fuzzy set* S is defined as a function $\mu_S(x)$ assigning to each object x a number $\mu_S(x) \in [0, 1]$.

2. Formulation of the Problem

- Zadeh's idea of capturing the fuzziness of human reasoning has led to numerous successful applications.
- Fuzzy techniques are used in control, in clustering, etc.
- However, it is not always easy to apply fuzzy techniques:
  - there are many alternative fuzzy techniques, and
  - it is not clear which of these techniques we should use.
- Indeed, fuzzy sets are usually defined as generalizations of crisp sets.
- There are many ways to extend a notion from crisp sets to fuzzy sets.
- Thus, there are many and-operations (t-norms), or-operations (t-conorms), etc.

3. Our Idea

- We show that fuzzy sets can be also naturally interpreted as *limits* of crisp sets.
- Once a fuzzy set S is represented as a limit of a sequence S_n of crisp sets, then we can define, e.g.,
  - the probability $P(S)$ of this fuzzy set as
  - a limit of the probabilities $P(S_n)$ of the corresponding crisp sets.
- We will show that in some cases, this idea indeed enables us to select one definition among many.
- We will also show that this idea is not a panacea: sometimes, there is no limit :-(

4. Polling Interpretation of Fuzzy Properties

- One of the standard ways to elicit the membership degrees $\mu_S(x)$ is by polling.
- We ask several (N) experts whether x satisfies the property P (e.g., "small").
- If M out of N folks say that x is small, we take $\mu_S(x) = M/N$.
- There are infinitely many values x. We can only ask experts about finitely many values, and take

$$\mu_S(x) \approx \frac{\#(C_n^+ \cap (x-\varepsilon, x+\varepsilon))}{\#(C_n \cap (x-\varepsilon, x+\varepsilon))}, \text{ where:}$$

  - C_n is the set of n queried values;
  - $C_n^+ \subseteq C_n$ is the set of all values marked as satisfying P.
- Ideally, we should tend to the limit $n \to \infty$ and $\varepsilon \to 0$.

5. From Finite Lists C_n to Crisp Sets S_n

- We have a list C_n of values which experts claim P or $\neg P$.
- To check whether $x \notin C_n$ satisfies P, it is reasonable to find the element $c \in C_n$ which is the closest to x:
  - If P holds for c, we conclude that P holds for x.
  - If $\neg P$ holds for c, we conclude that $\neg P$ holds for x.
- Example:
  - $C_n = \{0.1, 0.2, 0.4, 0.6, 0.7, 0.9, 1.3\}$, $C_n^+ = \{0.6, 0.9, 1.3\}$.
  - For $x = 0.62$, the closest is $c = 0.6$, so we conclude that $P(x)$.
  - For $x = 0.67$, the closest is $c = 0.7$, so we conclude that $\neg P(x)$.
  - The set S_n of all x for which $P(x)$ holds is $[0.5, 0.65] \cup [0.8, \infty)$.

6. Fuzzy Sets as Limits of Crisp Sets: Idea

- We have a sequence of crisp sets S_n.
- Based on each set S_n, we can approximate $\mu_S(x)$ as

$$\mu_S(x) \approx \frac{\text{len}(S_n \cap (x-\varepsilon, x+\varepsilon))}{\text{len}(x-\varepsilon, x+\varepsilon)}.$$

- In the limit, we get the exact value:

$$\mu_S(x) = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} \frac{\text{len}(S_n \cap (x-\varepsilon, x+\varepsilon))}{\text{len}(x-\varepsilon, x+\varepsilon)}.$$

- We can define a fuzzy set S as a limit of crisp sets S_n if the above formula holds.
- In the multi-D case:
  - we use boxes $(x_1-\varepsilon_1, x_1+\varepsilon_1) \times \ldots \times (x_d-\varepsilon_d, x_d+\varepsilon_d)$ instead of intervals; and
  - we use area, volume, etc., instead of length.

7. Every Fuzzy Set Can Be Represented as a Limit of Crisp Sets

- **Proposition.** *Every fuzzy set with a continuous membership f-n can be represented as a limit of crisp sets.*
- **Proof.** Let's divide the real axis into intervals $\left[\frac{k}{n}, \frac{k+1}{n}\right)$ corresponding to different integers k.
- We then divide each such interval into two parts:
  - a part $\left[\frac{k}{n}, \frac{k}{n} + \mu\left(\frac{k}{n}\right) \cdot \frac{1}{n}\right)$, a portion $\mu\left(\frac{k}{n}\right)$, is assigned to the set S_n, and
  - the remaining part $\left(\frac{k}{n} + \mu\left(\frac{k}{n}\right) \cdot \frac{1}{n}, \frac{k+1}{n}\right)$ is assigned to the complement of the set S_n.
- For the resulting sequence $S_n = \bigcup\limits_k \left[\frac{k}{n}, \frac{k}{n} + \mu\left(\frac{k}{n}\right) \cdot \frac{1}{n}\right)$, the limit equality holds for every x. Q.E.D.

8. The Limit Representation Enables Us to Uniquely Describe Probability of a Fuzzy Set

- Let $\rho(x)$ be a continuous probability density.
- For each crisp set s, we compute its probability as $P(s)=\int_s \rho(x)\,dx$.
- A real number $P(S)$ is called the *probability* of a fuzzy set S if for every sequence S_n with $S_n \to S$, we have

$$P(S_n)\to P(S).$$

- **Proposition.** *For every fuzzy set S with a continuous memb. f-n $\mu(x)$, $P(S)$ is well-defined and equals to*

$$P(S)=\int \mu(x)\cdot\rho(x)\,dx.$$

- This result provides one more justification for the original Zadeh's definition of the probability of a fuzzy set.

9. Another Case When the Limit Idea Leads to a Unique Generalization: Complement

- **Definition.** *We say that a fuzzy set S' is a* comple-ment *to a fuzzy set S if:*
  - *for every sequence S_n of crisp sets for which*

    $$S_n \to S,$$

  - *we have $-S_n \to S'$ for the sequence of their complements $-S_n$.*
- **Proposition.** *For every fuzzy set S with a continuous membership function $\mu(x)$,*
  - *its complement S' is well-defined, and*
  - *the membership function of S' is equal to*

    $$\mu_{S'}(x) = 1 - \mu_S(x).$$

10. The Limit Idea Is Not a Panacea

- A similar idea does not lead to unique union $\cup$ or intersection $\cap$ operations.
- *Idea:* if $S_n \to S$ and $S'_n \to S'$ imply $S_n \cup S'_n \to S \cup S'$.
- *Problem:* different sequences $S_n \to S$ and $S'_n \to S'$ lead to different limits for $S_n \cup S'_n$.
- *Example:* $\mu_S(x) = \mu_{S'}(x) = 0.5$ for all $x \in [0,1]$.
- We can take $S_n = S'_n = \bigcup\limits_k \left[\frac{k}{n}, \frac{k}{n} + \frac{1}{2} \cdot \frac{1}{n}\right)$, then $S_n \cup S'_n = S_n$, and thus, in the limit, $\mu(x) = 0.5$.
- Alternatively, we can take $S'_n = \bigcup\limits_k \left[\frac{k}{n} + \frac{1}{2} \cdot \frac{1}{n}, \frac{k+1}{n}\right)$, then $S_n \cup S'_n = [0,1]$ and, in the limit, $\mu(x) = 1$.
- For intersection, same sequences of sets lead to $\mu(x) = 0.5$ and $\mu(x) = 0$.

11. Discussion

- We described how a fuzzy set can be represented in terms of several crisp sets – namely, as a *limit*.
- Another known way of representing a fuzzy set in terms of crisp sets is a representation in terms of alpha-cuts

$$\{x : \mu_S(x) \geq \alpha\}.$$

- The main difference is that
  - to uniquely determine a fuzzy set, we need to know *all* its alpha-cuts,
  - while we do not need all limit sets S_n to uniquely determine a fuzzy set;
  - it is sufficient to know the sets S_{n_k} for some sequence $n_k \to \infty$ (e.g., for $n_k = k^2$).

12. Future Work

- We use the property that $S_n \to S$ implies

$$P(S_n) \to P(S).$$

- In mathematical terms, the property is known as *continuity*.
- In these terms, we can say that:
  - probability is a continuous function of sets (in the sense of convergence $S_n \to S$);
  - complement is a continuous operation, while
  - union and intersection are *discontinuous* operations.
- For such discontinuous operations, instead of a single limit value, we have an *interval* of possible limit values.
- So maybe we can extend the limit idea to interval-valued fuzzy sets?

13. Acknowledgments

This work was supported in part:

- by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence),
- by the National Science Foundation grant DUE-0926721,
- by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and
- by grant N62909-12-1-7039 from the Office of Naval Research.